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THE DUAL RECIPROCITY BOUNDARY ELEMENT METHOD AND NUMERICAL SOLUTIONS OF ONE-DIMENSIONAL TRANSIENT HEAT TRANSFER PROBLEM

WALTER ARTHUR NONDI*

THOMAS ONYANGO**

S. N. MUTHIGA*

N. B. Okelo***

Abstract

The dual-reciprocity boundary element method for the solution of a one-dimensional transient heat equation with a source term is considered. We give a numerical solution to this kind of heat equation. Discretization of heat equation using BEM and the solution of boundary integral equation is further discussed and results presented in graphs.

* Catholic University of Eastern Africa, Kenya.

** Technical University of Kenya, Kenya.

*** School of mathematics and actuarial science, Jaramogi Oginga Odinga University of Science and Technology

Introduction

We consider a one-dimensional heat conduction equation which models an unsteady temperature distribution in a solid (domain Ω). This problem is governed by the differential equation

$$k\nabla^2 T(x,t) + Q(x,t) = \rho c \frac{\partial T(x,t)}{\partial t} \quad \text{……………………………}(3.46)$$

where $Q(x,t)$ is the source term.

with the Neumann boundary conditions;

$$q(x,t) = \bar{q}(x,t) \qquad x \in \Gamma_q \quad \text{……………………………………….}(3.47)$$

And the initial condition

$$T(x,0) = T_0 \qquad T \epsilon\, \Omega \quad \text{……………………………………..…}(3.48)$$

Time-Stepping Scheme

This scheme is used to handle the time-variable of heat conduction process and then the system is replaced by a set of inhomogeneous modified Helmholtz equations. There are different approaches to handle time variable, two of which are; (1) Laplace transform; (2) Finite differencing in time. Since numerical inversion of the Laplace transform is often ill-poised, here we apply the finite difference scheme to handle the time variable. For a typical time interval $[t^n, t^{n+1}] \in [0,T]$, $T(x,t)$, its derivative with respect to time variable t and $Q(x,t)$ are approximated as

$$T(x,t) = \theta T^{n+1} + (1-\theta)T^n(x) \quad \text{…………………………..…………}(3.49)$$

$$\frac{\partial T(x,t)}{\partial t} = \frac{T^{n+1}(x) - T^n(x)}{\tau} \quad \text{… ……………………………………..…………..}(3.50)$$

$$Q(x,t) = \theta Q^{n+1}(x) + (1-\theta)Q^n(x) \quad \text{……………………………….…...}(3.51)$$

where the superscripts n and $n+1$ refer to subsequent time instances and $\tau = t^{n+1} - t^n$ is the step size, θ $(0 \le \theta \le 1)$ is a real parameter that determines if the method is explicit $(\theta = 0)$, implicit $(\theta = 1)$or a linear combination of both. It is easily verified that the conditions which prevent oscillations in explicit case are exactly the same as the commonly cited sufficient conditions which ensure that it is stable. Furthermore, even though a Crank Nicolson approach is unconditionally stable, it permits the development of spurious oscillations unless the step size is no more than twice that required for an explicit method to be stable. Although an explicit scheme is only first-order accurate in time, it is proved that the PDE can be solved accurately using the implicit scheme. Hence, we use $\theta = 1$ in our analysis. Substituting equation (3.50) into equation (3.51) and rearranging it yields the following modified Helmholtz-type equation that has to be solved at each time step t^{n+1} for the unknown $T^{n+1}(x)$;

$$\nabla^2 T^{n+1}(x) - \frac{\rho c}{k\tau} T^{n+1}(x) = -\frac{\rho c}{k\tau} T^n(x) - \frac{1}{k} Q^n(x) \quad \text{………………………..}(3.52)$$

Note that the right-hand side of equation (3.52) is well defined in terms of approximate solution T^n calculated on the previous time step $t = t^n$.To start the procedure we take $T(x,0) = T_0$,the initial condition of the transient problem. For simplicity, the single step formula of equation (3.52) can be written as;

$$(\nabla^2 - \lambda^2)T(x) = f(x) \quad \ldots(3.53)$$

where

$$\lambda = \sqrt{\frac{\rho c}{\tau k}}$$

$$f(x) = -\frac{\rho c}{\tau k}T^n(x) - \frac{Q^n(x)}{k} \quad \ldots(3.54)$$

Equation (3.54) is a sequence of inhomogeneous modified Helmholtz equation

Dual Reciprocity Boundary Element Method (DRBEM) for particular Solution

The DRBEM employs a fundamental solution corresponding to a simpler equation, and treats the remaining terms as well as other non-homogeneous terms in the original equation, through a procedure which involves a series expansion using global approximating functions and the applications of the reciprocity principle (Partridge et.al, 1992).The method is called dual reciprocity method because it utilizes twice times of basic reciprocal theorem. It is essentially a method of constructing particular solutions that can be used to solve non-linear and time-dependent problems as well as to represent any internal source distribution. The main idea of the DRBEM is to divide the solution into two parts; a known particular solution of the inhomogeneous plus a complementary solution of its inhomogeneous counterpart. Since particular solutions to complex problems are very difficult or sometimes even impossible to obtain, the in homogeneity is approximated by a series of simpler radial basis functions (RBFs) for which particular solutions can be easily determined. In particular, the DRBEM, which transforms domain integrals to boundary integrals by combining radial basis functions and conventional BEM, has wide applications in practical engineering. Due to linear property of equation (3.53), its solution can be expressed as a summation of a particular solution T_p and a homogeneous solutionT_h

$$T = T_p + T_h$$

where T_p satisfies the inhomogeneous equation

$$(\nabla^2 - \lambda^2)T_p(x) = f(x) x \epsilon \Omega \quad \ldots(3.55)$$

but does not necessarily satisfy the boundary condition.

$T_h$satisfies,

$$(\nabla^2 - \lambda^2)T_h(x) = 0, \quad x \in \Omega \quad \ldots(3.56)$$

The particular solution T_p can be obtained by DRBEM. To do this the right-hand side term of equation (3.55) is approximated by RBF, yielding

$f(x) = \sum_{i=1}^{N} \alpha_i \varphi_i(x)\, x \in \Omega$.. (3.57)

where N is the number of interpolation points in the domain under consideration. Here,

$\varphi_i(x) = \varphi(r) = \varphi(|x - x_i|)$ denotes radial basis functions with the reference point x_i and α_i are interpolating coefficients to be determined.

Simultaneously, the particular solution T_p is similarly expressed as

$T_p(x) = \sum_{i=1}^{N} \alpha_i \psi_i(x)$..(3.58)

where ψ_i represents corresponding approximated particular solutions which satisfy the following differential equations.

$(\nabla^2 - \lambda^2)\psi_i = \psi_i$..(3.59)

noting the relation between the particular solution T_p and the function $f(x)$ in equation (3.57).

By enforcing equation (3.59) to satisfy equation (3.58) at all nodes, we can obtain a set of simultaneous equations to uniquely determine the unknown coefficients α_i.

These coefficients can be evaluated numerically.

Numerical Solutions

In this section, two numerical examples of heat conduction problems, to which analytical solutions are available, are used to test the accuracy and efficiency of DRBEM.

Example 1

Consider heat equation

$U_t = U_{xx} + 2 \qquad 0 < x < 1,\ \ 1 < t < 2$...........................(4.1.0)

The boundary conditions are;

$U_x(o,t) = 0 \qquad 1 < t < 2$.......................................(4.2.0)

$U(1,t) = 1 + 4t 1 < t < 2$....................................... (4.3.0)

The initial condition is;

$U(x,0) = x^2 0 < x < 1$....................................... (4.4.0)

We solve the equation using the DRBEM and compare the results with the analytic solution.

The exact solution to the above heat equation is

$U(x,t) = x^2 + 4t$ .... (4.5.0)

Figure 2: Temperature distribution against x, when N=5

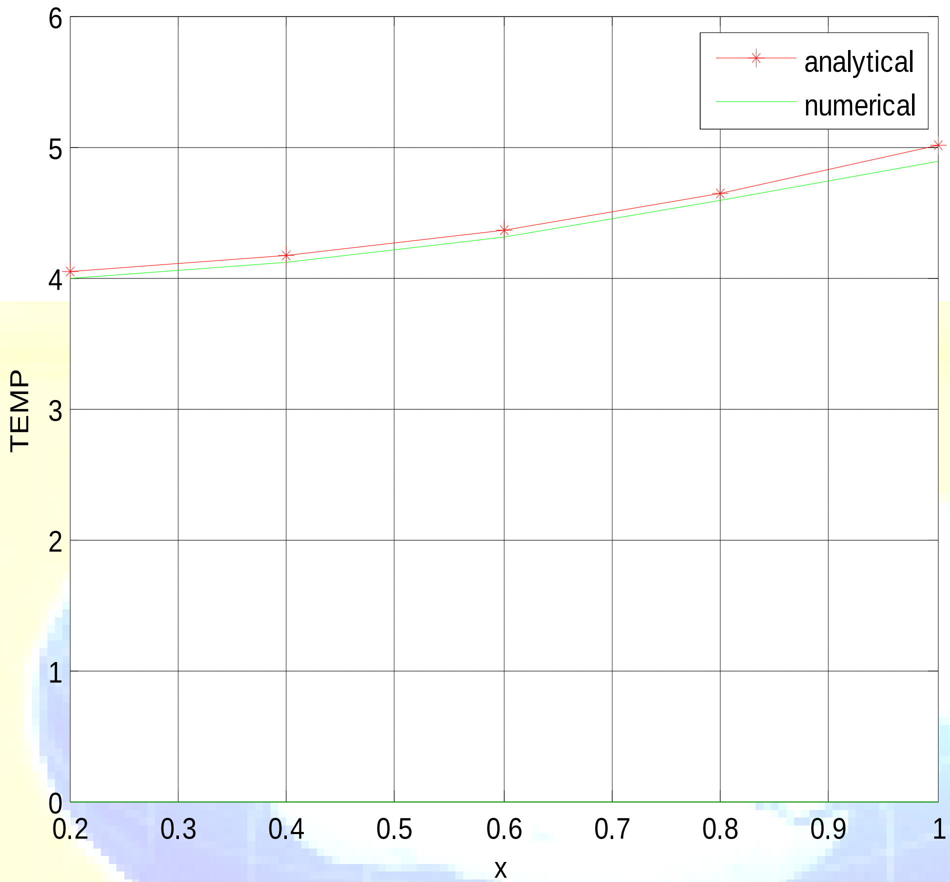
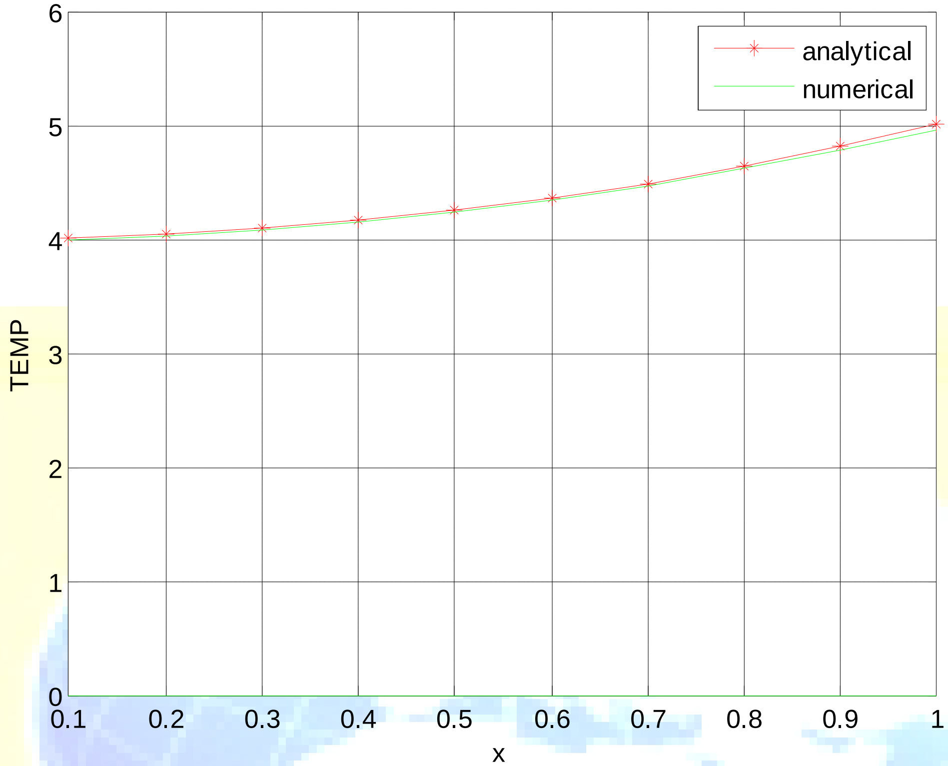


Figure 3; Temperature distribution against x, when N=10



Example 2

Consider the heat conduction problem

$U_t(x,t) = U_{xx}(x,t) + f(x)$;$0 < x < 1, 1 < t < 2$,……………(4.2.0)

The boundary conditions are;

$U(0,t) = 2$ $\quad 1 < t < 2$ (4.2.1)

$U(1,t) = 3$ $\quad 1 < t < 2$ (4.2.2)

The initial condition is;

$U(x,0) = x^2 + \sin(2\pi x)$; $\quad 0 \le x \le 1$(4.2.1)

The problem has a dirichlet homogeneous boundary surface at one end. The exact solution of this problem is

$U(x,t) = x^2 + 2t + \sin(2\pi x)$; $\quad 0 \le x \le 1, 1 \le t \le 2$………(4.2.2)

$f(x) = 4\pi^2 \sin(2\pi x)$…………………...(4.2.)

Figure 4; Temperature distribution against x, when the time step is 1, time interval is 0.04, with N=50.

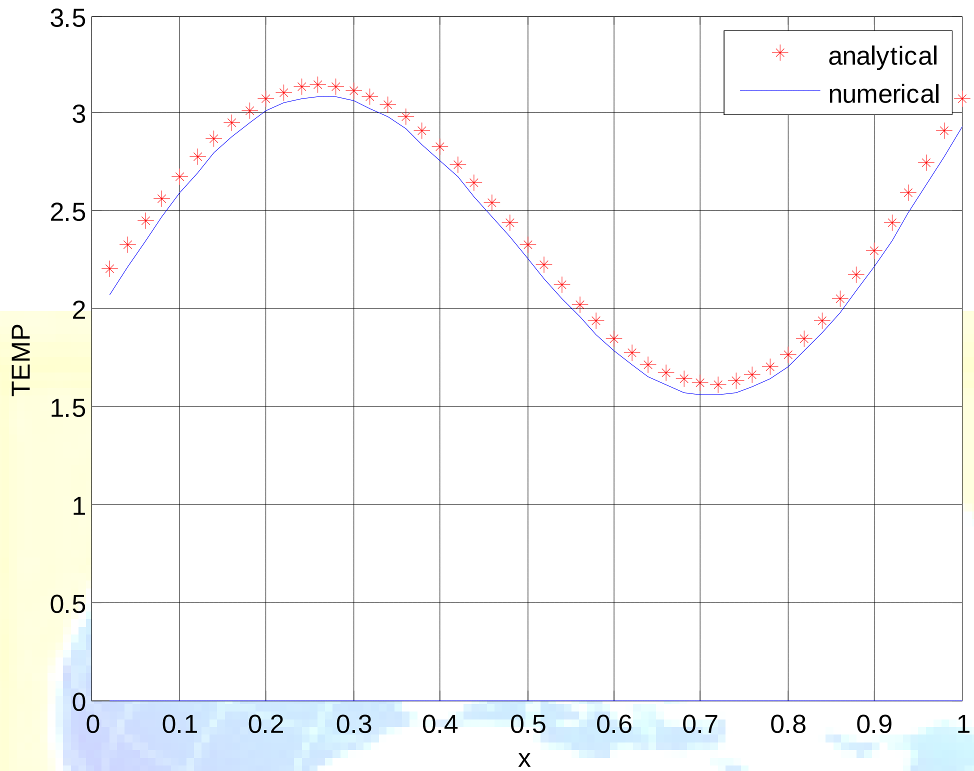


Figure 5; Temperature distribution when the time step is 1, time interval is 0.02, with N=50

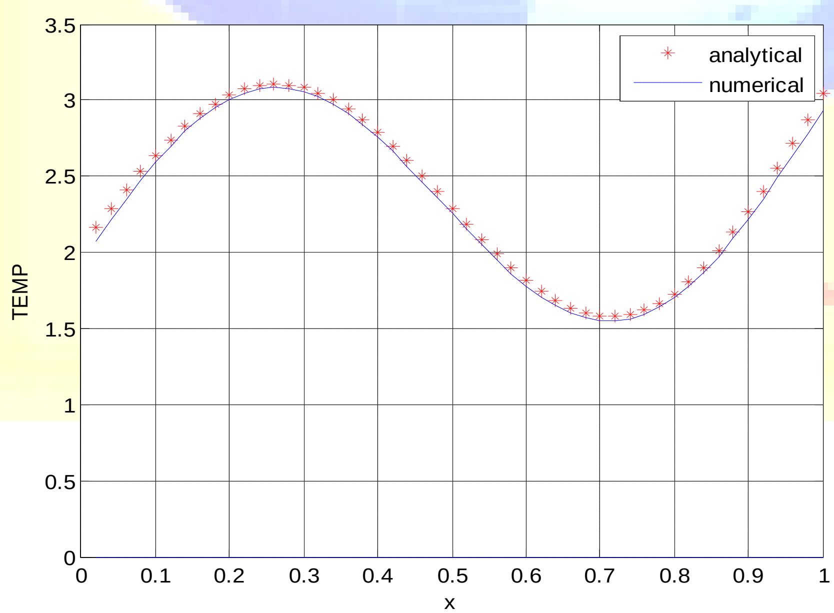
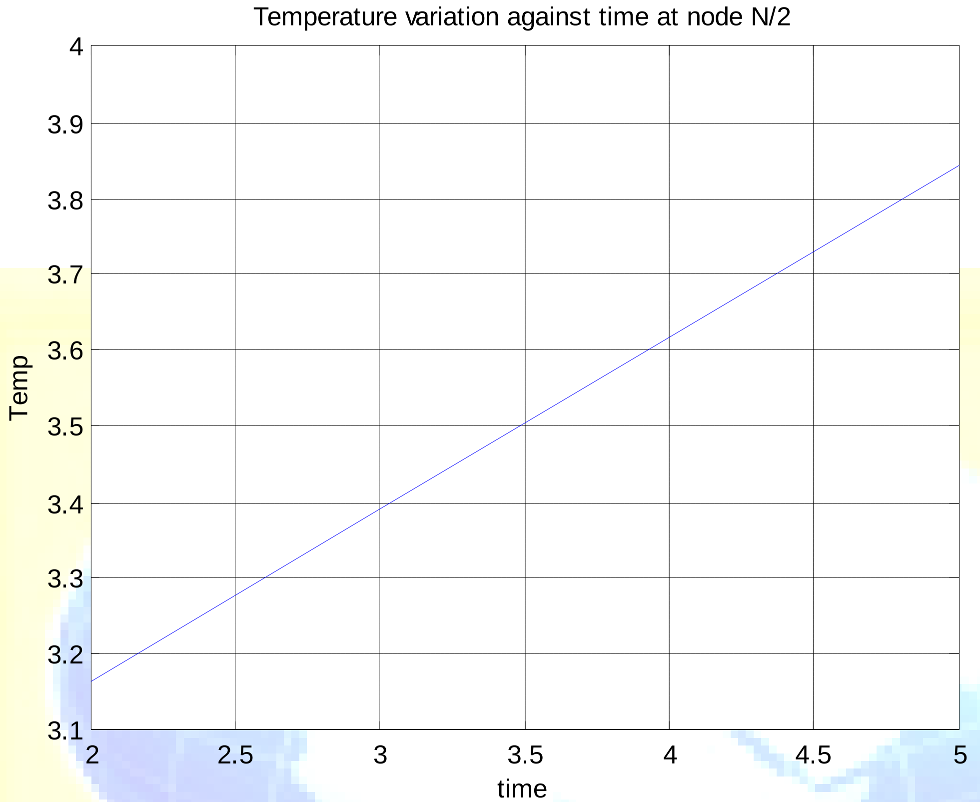


Figure 6; Temperature variation against time at the center node.



Discussion of graphs

Figure 2, shows the temperature distribution against space variable x when the nodes are five. The source term in the heat equation is two. At temperature $t > 0$, heat increases within the bar because of the source term. As more nodes are increased the flow becomes smooth. The temperature distribution along the bar increases. The analytic solution and the numerical solution coincide as the discretization nodes are increased.

Figure 4 and figure 5 show the temperature distribution against spacial variable x along the rod. This is a sinusoidal graph that stabilizes as the number of time step is reduced. It clearly shows when the time interval at time step size of one is reduced the analytical and numerical solutions converge. Thus, justifying the DRBEM as a numerical solution that converges as the discretization nodes are increased.

Figure 6, shows the temperature variation against time in the center node. Temperature increases linearly with time, an indication that within the bar there is an internal heat generating term, which is the source term.

These graphical results shows the importance of source term in the design of materials used in heat generating regions. The DRBEM can be developed as an efficient method to model and analyze the heat and cooling systems which are governed by the heat equations. For example, the combustion chamber and nozzle walls have to withstand relatively high temperature, high gas velocity, chemical erosion and high stress. The choice of materials used should be such that the wall material must be capable of enduring high heat transfer rates (which means good thermal conductivity) yet, at the same time, have adequate strength to withstand the chamber combustion pressure. All these considerations are as a result of the presence of the source term. Material requirements are critical only in those parts which come into direct contact with propellant gases.

Conclusion

The main objective of the study was to solve a one-dimensional transient heat conduction problem using DRBEM. With the help of the governing equations in chapter three fundamental solutions for heat equation was discretized over the time and space. A thin finite slab geometry $(0,1)\times(0,1)$ was used with specified boundary conditions to model the problem. The mathematical software MATLAB was used to compute results and graphs were drawn at various nodes of discretization. Two test problems, parabolic and sinusoidal functions were used. It is clear that the comparison of the analytical and numerical results presented by the DRBEM leads to a rapid convergence to the exact solutions with small discretization of time parameter. The results obtained from both problems were very close to the analytical solution confirming the efficiency of DRBEM. The effect of increasing the number of discretization of the boundary resulted high convergence. It is clearly shown that the presence of internal heat generation or source term leads to an increase in temperature distribution within the material. However, this increase depends on the function of the source term. These findings can help in the design of thermal insulators in high heat generating materials and also in choosing the type of materials to use in making conductors and insulators.

Recommendation

For future work, we may extend the DRBEM formulation to transient problems governed by convection-diffusion equations with a constant or variable velocity field. In addition, we may try to investigate the feasibility of applying the DRBEM to non-linear scalar wave propagation problems or hyperbolic heat conduction problems. From the mathematical point of view, more theoretical studies are required in order to support numerical studies related to convergence of the solution of a linearized system to the true solution of the original nonlinear differential system. However, this could be quite involving but well rewarding.

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