MAASAI MARA UNIVERSITY

REGULAR UNIVERSITY EXAMINATIONS

2018/2019 ACADEMIC YEAR

THIRD YEAR SECOND SEMESTER

SCHOOL OF SCIENCE & INFORMATION SCIENCES

BACHELOR OF SCIENCE -MATHEMATICS

COURSE CODE: MAT 3224

COURSE TITLE: INTRODUCTION TO MATHEMATICAL MODELING

DATE: TUESDAY 23RD APRIL 2019 TIME: 0830-1030HOURS

INSTRUCTIONS TO CANDIDATES

Answer Question **ONE** and any other **TWO** questions

This paper consists of ***6*** *printed pages. Please turn over.*

SECTION A (COMPULSORY)

QUESTION ONE (30 marks)

a) Consider the differential equation $\dot{x} = 2x(1-\frac{x}{2})(x-1)$.

i) Draw the phase line for the differential equation and classify the equilibrium points as sinks, sources and nodes.

4mks

ii) Give a rough sketch of the slope field for this differential equation and draw a few solutions into the slope field.

2mks

iii) Consider the solution to the differential equation which satisfies the initial condition

2mks

$x(0)=1.5$, find $\lim_{t\to\infty} x(t)$

$x(0)=3,$ Find $\lim_{t\to\infty} x(t)$

iv) Show that $\bar{x}=2$ is stable while $\bar{x}=1$ is unstable given that

$f(x) = 2x(1-\frac{x}{2})(1-x).$

4mks

b) Let $N(t)$ be the population of a species at time $t,$ then the rate of change of population is

$\dot{N} = births - deaths + migration$................................$*$ Assuming that there is no migration, b and d are the rates of births and deaths respectively. Write down equation * in terms of $N(t), b$ and d.

2mks

c) Solve the equation in (b) above and sketch on the same diagram the solution curves for $b > d, b < d, b = d.$
6mks

d) Consider the recurrence relation $N_{t+1} = rN_t(1 - \frac{N_t}{K}).$

i) Normalize the relation and find the expression for $f^2(u_t, r)$.
3mks

ii) Determine the non-zero fixed points of $f(u_t, r)$ and their stability. **7mks**

QUESTION TWO (20 marks)

The population of a species is governed by the discrete model

$N_{t+1} = f(N_t) = N_t \exp\{r(1 - \frac{N}{K})\}$ where r and K are positive constants.

a) Determine the steady states and their eigenvalues.
4mks

b) Find the expression for the maximum population N_M.
4mks

c) Find the expression for the minimum population N_m.
4mks

d) A population will become extinct if $N_t < 1$. Show that the condition for extinction for the population is $K \exp[2r - 1 - e^{r-1}] < r.$ **4mks**

e) Sketch the curve for N_{t+1} against N_t for the expression in (a). **4mks**

QUESTION THREE (20marks)

The director of Kenya Wildlife Service is planning to issue antelope hunting permits to Nairobi National park. The director knows that if the antelope population falls below a certain level $m>0,$ the antelope will become extinct. The director also knows that the National park has a maximum carrying capacity M, so that if the population goes above m, then it will increase to M. A simple model for the population growth in the park is found to be $\dot{N}=\kappa N(M-N)(N-m)=: f(N)$ where $N:=N(t)$ is the antelope population at time $t,$ and $\kappa>0$ is a constant.

a) Find all the fixed points of the population.
3mks

b) Determine their nature of stability.
2mks

c) Draw the phase portrait of the equation above, that is the curve of $f(N)$ against N and indicate the nature of flow determined by the fixed points in (a) and (b) above.
7mks

d) Investigate and describe qualitatively the change in population with time for the current antelope population $N_0>0$ in three cases namely:

i) $N_0<m$
2mks

ii) $m<N_0<M$
3mks

iii) $N_0>M$
3mks

QUESTION FOUR (20 marks)

We are interested in the development of a simple AIDS epidemic model in heterosexual population of adults. Let a population be divided into three categories; $S(t), I(t), A(t)$ as defined below.

$S(t)$: Susceptibles, the number of individuals at the time t, not yet infected but
May be infected if exposed to the disease.

$I(t)$: Infectives, the number of individuals at time t, who are already
Infected with HIV/AIDS and are capable of transmitting the virus.

$A(t)$: The number of individuals who have developed full-blown AIDS
Symptoms at time t.

μ: The per capita AIDs nonrelated mortality rate .

d: The rate at which AIDs patients are dying due to AIDs causes.

υ: The rate at which HIV infected (infectives) progress to AIDs

λ: The probability of getting infected by HIV/AIDs from a randomly chosen
Partner.

c: The rate at which an individual acquires new (changes) sexual partner.

β: The transmission probability that is the probability of getting infected
From a partner.

B: Recruitment rate of susceptibles into a population.

We make the following assumptions

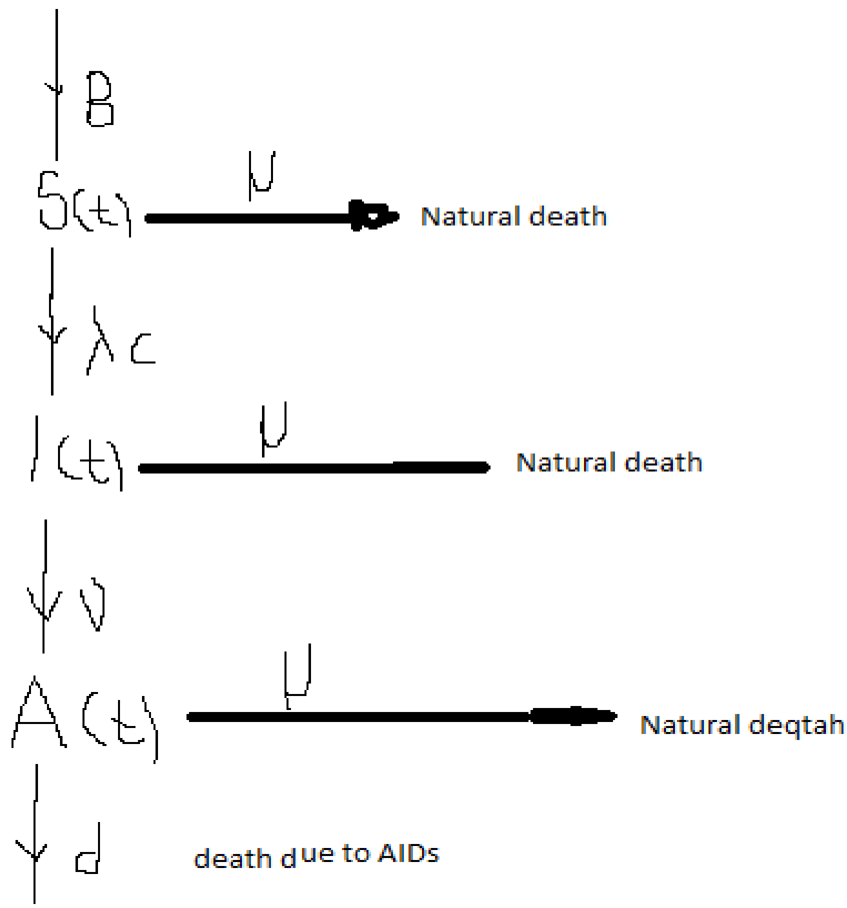
i) The recruitment into the population of study (sexually mature adults) is mainly by birth.

ii) The full blown AIDs cases are easily recognized in the population and are nolonger a threat in the spread of the epidemic: that is they do not participate in the population dynamics.

iii) An individual once infected becomes and remains infective until death.

iv) The force of infection depends on the number of infectives in the population and the product βc.

v) We consider a homogenous population(uniform mixture).



A reasonable first model based on the flow diagram is

$$\dot{S} = B - \mu S - \lambda c S, \lambda = \frac{\beta I}{N} \text{..............} 1$$

$$\dot{I} = \lambda c S - (\mu + \upsilon) I \text{............................} 2$$

$$\dot{A} = \upsilon I - (d + \mu) A \text{.............................} 3$$

Where $\frac{1}{\upsilon}$ a constant is the average incubation time of the disease and $N(t) = S(t) + I(t)$.

a)(i)What is the interpretation of $\frac{I}{S+I}$

2mks

ii) If an individual is full blown AIDs dies within 9 months to 12 months, state the interval of existence of the parameter d.
2mks

iii)If the incubation period is 8 months, what is the value of υ.
1mk

iv) From equation 2, write down an expression for the basic reproductive rate of the infection R_0.
2mks

b) In equation 2, if at t=0, an infected individual is introduced into an otherwise infection free population of susceptibles, we have initially $S \approx N$. Since the average incubation time from infection to development of the disease is very much shorter than the average life expectancy of the susceptible, that is $\upsilon ? \ \mu$, we have that near t=0 $\dot{I} = (\beta c - \upsilon - \mu) I \approx \upsilon (R_0 - 1) I$..........4

i) What is the expression for the reproductive rate of the infection R_0. Described in (4).
2mks

ii) Write an expression for the solution of equation (4) if the initial population
is $I(0) = I_0$.
2mks

iii)From the solution in (ii) above, determine an expression for the doubling
time of the infectives.
2mks

c) We notice that R_0 depends on β, c both of which are social factors. What could one do to keep R_0 small and hence lower the rate of increase of
$I(t)$.
2mks

d) Suppose that once an individual is tested HIV positive is exposed and thus avoided sexually, what modification could one introduce to equation (2) to carter for the variation.
2mks

e)Suppose the life expectancy of children is $\frac{1}{\mu}$, what proportion of children born $\tau > 0$ years ago can reach sexual maturity age τ years**.** **3mks**

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