MAASAI MARA UNIVERSITY

MAIN EXAMINATION 2018/2019 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER EXAMINATIONS

FOR

THE DEGREE OF BACHELOR SCIENCE IN MATHEMATICS

MAT 416: FUNCTIONAL ANALYSIS I

DATE: 26TH APRIL 2019 TIME: 0830 – 1030 HRS

INSTRUCTIONS TO CANDIDATES

1. This paper contains **FOUR** (4) questions
2. Answer question **ONE (1)** and any other **TWO** (**2**) questions
3. Do not forget to write your Registration Number.

QUESTION ONE (30MARKS)

a) Define the following terms

i) A Banach space
1mark

ii) Strongly convergence of a sequence
1mark

iii) A Hilbert space
1 mark

iv) Radius of convergence of a series
1 mark

b) Show that an integral operator is a bounded linear transformation. **5marks**

c) Show that $\left\{ \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right), \left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right) \right\}$ is an orthonormal set

5marks

d) Define a normed linear space and show that if X is an inner product space ,

then $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$ defines a norm on X **5marks**

e) In the polynomial space p^2 the inner product is given as

$\langle u,h\rangle = \int_0^1 u(t)\,h(t)\,dt$. if $u(t) = t+2$ and $h(t) = t^2 - 2t + 3$. Find

i. $\langle u,h\rangle$ ii. $\|u\|$ iii. $\|h\|$
8marks

e) Given that $x = \sum_{\alpha \in \Lambda} \left| \langle x, z_\alpha \rangle z_\alpha \right|$ $\forall\ x \in H$. Show that $\|x\|^2 = \sum_{\alpha \in \Lambda} \left| \langle x, z_\alpha \rangle \right|^2$ **3marks**

QUESTION TWO (20MARKS)

a) Show that the differential operator $T : C_{[a,b]} \to C_{[a,b]}$ defined by $Tx(t) = x'(t)$ is an unbounded linear transformation
5marks

b) State and prove the Reisz representation theorem
10marks

c) Let $a \in \mathbb{R}^3$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x) = \langle x, a \rangle$ for all $x \in \mathbb{R}^3$ **.** Show that f is a bounded linear functional with $\|f\| = \|a\|$
5marks

QUESTION THREE (20MARKS)

a) Define bounded linear transformation.
3marks

b) Show that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$
3marks

c) If $(\langle . \rangle, X)$ is an inner product space, show that for all $x, y \in X$ we have $\left| \langle x, y \rangle \right|^2 \le \langle x, y \rangle \langle x, y \rangle$
7marks

d) Show that E is closed with respect to the Hilbert space H if and only if it is a complete orthonormal subset
7marks

QUESTION FOUR (20MARKS)

a) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n\,x^n}{2^{n+1}}$$

5marks

b) State and prove the projection theorem
6marks

c) Show that u for all $f, g \in L_2(a,b)$ the $\langle f, g\rangle = \int_a^b f(x)\,\overline{g}(x)\,dx$ defines an inner product on $L_2(a,b)$.
4marks

d) Suppose X and Y are Banach spaces and that T is a bounded linear operator from X to Y **.** If T maps X on to Y**,** show that $T(G)$ is open in Y whenever G is open in X.
5marks

//END