UNIVERSITY EXAMINATIONS, 2018
FOURTH YEAR EXAMINATION FOR
THE DEGREE OF BACHELOR OF MATHEMATICS
MAT:418- PARTIAL DIFFERENTIAL EQUATIONS I
Instructions to candidates:
Answer Question 1 and any other TWO.
All Symbols have their usual meaning

DATE: 2018 TIME: 2hrs

Question 1(Entire course: 30 Marks)

(a) A thin bar located on the x axis has its ends at $x = 0$ and $x = L$. The initial temperature of the bar is $f(x)$, $0 < x < L$, and its ends $x = 0, x = L$ are maintained at constant temperatures u_1, u_2 respectively.

(i) Assuming the surrounding medium is at temperature u_0 and that Newton's law of cooling applies, show that the partial differential equation for the temperature of the bar at any point at any time is given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \beta(u - u_0), \quad 0 < x < L, \quad t > 0, \tag{1}$$

(**4 Marks**)

(ii) Determine the steady state temperature of Equation (1) (**3 Marks**)

(b) **(Wave Equation)**

(i) Solve the initial value problem:

$$\begin{aligned} u_{tt} - \Delta u &= xt \quad , \text{ in } \mathbb{R}, t > 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= 0 \end{aligned}$$

(**4 Marks**)

(ii) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F, G. (**3 Marks**)

(iii) Using the change of variables $\xi = x + t, \quad \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$. (**4 Marks**)

(c) Consider the Partial differential equation

$$\begin{aligned} u_{x_1} + u_{x_2} &= u^2 \quad \text{in} \quad U \subseteq \mathbb{R}^2, \\ u &= g \quad \text{on } \Gamma, \text{ the boundary of } U. \end{aligned}$$

where $U = \{x_2 > 0\}$ and $\Gamma = \{x_2 = 0\} = \partial U$.

(i) Sketch the region U and indicate its boundary Γ. (**1 Marks**)

(ii) Find the Characteristic equations for (2) (**3 Marks**)

(ii) Find the initial Condition in parametric form (**5 Marks**)

(iv) Use the Characteristic equations and the initial conditions in (c)(ii) to find the solution

$$u = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}.$$

(3 Marks)

Question 2: Heat Equation, eigenfunction expansion (20 Marks)

Consider the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < L, \quad t > 0, \tag{2}$$

where $u := u(x,t), \alpha$ is a real constant, $f(x,t)$ a given function and L is a given constant. Suppose Equation(2) is to be solved subject to:

$$BCs \;\; \text{(i)} \;\; u(0,t) = 0, \text{(ii)} \;\; u(L,t) = 0, \quad t > 0 \tag{3}$$

and

$$\text{IC} \;\; u(x,0) = \varphi(x), \quad 0 \le x \le L, \tag{4}$$

where $\varphi(x)$ is a given function.

To solve (2) we seek a nontrivial separable series solution of the form

$$u(t,x) := \sum_{n=1}^{\infty} T_n(t) X_n(x), \tag{5}$$

where $X_n(x)$ is a function of x alone that we find by solving the homogeneous equation associated with Equation(2) subject to boundary conditions (3), while $T_n(t)$ is a function of t alone found by solving a sequence of ODEs.

(a) Determine the eigenfunction $X_n(x)$. **(7 Marks)**

(b) Suppose we expand $f(x,t)$ thus:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x). \tag{6}$$

Determine an expression for $f_n(t)$. **(3 Marks)**

(c) Using Equation (5) and Equation(6) in Equation(2), show that the function $T_n(t)$ satisfies the first order ODE given by

$$\dot{T}_n(t) + (\alpha n \pi / L)^2 T_n(t) = f_n(t), \tag{7}$$

where the dot denotes differention with respect to time t. **(4 Marks)**

(c) Determine the initial condition, $T_n(0)$, to be imposed on Equation(7) and hence solve it. **(6 Marks)**

Question 3 (Nonlinear First Order-Method of Characteristics: 20 Marks)

Consider the Partial differential equation

$$\begin{aligned} u_{x_1}u_{x_2} &= u \quad \text{in} \quad U \subseteq \mathbb{R}^2, \\ u &= x_1^2 \quad \text{on } \Gamma, \text{ the boundary of } U. \end{aligned}$$

where $U = \{x_2 > 0\}$ and $\Gamma = \{x_2 = 0\} = \partial U$.

(a) Sketch the region U and indicate its boundary Γ. **(3 Marks)**

(b) Find the Characteristic equations for (8) **(5 Marks)**

(c) Find the initial Condition in parametric form **(5 Marks)**

(d) Use the Characteristic equations and the initial conditions in (c) to find the solution

$$u = \frac{(4x_1 + x_2)^2}{16}.$$

(7 Marks)

Question 4 : Transport Equation (20 Marks)

(a) The the one dimensional transport equation in all of $\mathbb{R}$ is given by

$$u_t + bu_x = 0 \quad \text{for} \quad x \in \mathbb{R}, t > 0, \tag{8}$$

where b is a constant. Show that $z(s) := u(x + sb, t + s), \quad s \in \mathbb{R}$ is a solution to Equation(8) and give its geometrical interpretation. **(5 Marks)**

(b) Suppose that Equation (8) is subject to the initial condition

$$u(x, 0) = g(x) \quad x \in \mathbb{R}. \tag{9}$$

Show that the solution to the transport equation (8) subject to (9) is

$$u(x, t) = g(x - tb)$$

(5 Marks)

(c) Consider

$$\begin{aligned} u_t + bu_x &= f \quad \text{for} \quad x \in \mathbb{R}, t > 0, \\ u(x, 0) &= g(x), \quad x \in \mathbb{R}. \end{aligned}$$

Derive the solution

$$u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)ds \tag{10}$$

of Equation(10). **(10 Marks)**