5.6 (*CD-ROM TOPIC*) USING THE POISSON DISTRIBUTION TO APPROXIMATE THE BINOMIAL DISTRIBUTION

You can use the Poisson distribution to approximate the binomial distribution when n is large and p is very small. The larger the n and the smaller the p, the better the approximation. The following mathematical expression for the Poisson model is used to approximate the true (binomial) result:

$$P(X) \cong \frac{e^{-np}(np)^x}{X!} \qquad \textbf{(5.18)}$$

where $P(X)$ = probability of X successes given the parameters n and p

n = sample size

p = probability of success

e = mathematical constant approximated by 2.71828

X = number of successes in the sample ($X = 0, 1, 2, \ldots n$)

The Poisson random variable theoretically ranges from 0 to ∞. However, when you use the Poisson distribution as an approximation to the binomial distribution, the Poisson random variable —the number of successes out of n observations—cannot be greater than the sample size n. With large n and small p, Equation (5.18) implies that the probability of observing a large number of successes becomes small and approaches zero quite rapidly.

As mentioned previously, in the Poisson distribution the mean μ and the variance σ^2 are each equal to λ. Thus, when you use the Poisson distribution to approximate the binomial distribution, Equation (5.19) is used to approximate the mean.

$$\mu = E(X) = \lambda = np \qquad \textbf{(5.19)}$$

Equation (5.20) is used to approximate the standard deviation.

$$\sigma = \sqrt{\lambda} = \sqrt{np} \qquad \textbf{(5.20)}$$

The standard deviation given by Equation (5.20) closely approximates the standard deviation for the binomial model [Equation (5.13) on page 172] when p is close to zero so that $(1 - p)$ is close to one.

Suppose 8% of the tires manufactured at a particular plant are defective. To illustrate the use of the Poisson approximation for the binomial, you calculate the probability of exactly one defective tire from a sample of 20 using Equation (5.18) as follows:

$$P(X = 1) \cong \frac{e^{-(20)(0.08)}[(20)(0.08)]^1}{1!} = \frac{e^{-1.6}(1.6)^1}{1!} = 0.3230$$

Alternatively, you can use tables of the Poisson distribution (Table E.7). Referring to these tables, the only values necessary are the parameter λ and the desired number of successes X. Since in the above example $\lambda = 1.6$ and $X = 1$, using Table E.7

$$P(X = 1) = 0.3230$$

This probability is shown in Table 5.7 (which is a portion of Table E.7).

TABLE 5.7
Finding a Poisson Probability

					λ					
X	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**
0	.3329	.3012	.2725	.2466	.2231	.2019	.1827	.1653	.1496	.1353
1	.3662	.3614	.3543	.3452	.3347	.3230	.3106	.2975	.2842	.2707
2	.2014	.2169	.2303	.2417	.2510	.2584	.2640	.2678	.2700	.2707
3	.0738	.0867	.0998	.1128	.1255	.1378	.1496	.1607	.1710	.1804
4	.0203	.0260	.0324	.0395	.0471	.0551	.0636	.0723	.0812	.0902

Source: Extracted from Table E.7.

Had the true distribution, the binomial, been used instead of the approximation,

$$P(X = 1) = \frac{20!}{1!19!}(0.08)^1(0.92)^{19} = 0.3282$$

This computation, though, is tedious. Clearly, with binomial tables such as Table E.6 or Microsoft Excel or Minitab available, you could find the binomial probability directly for $n = 20$, $p = 0.08$, and $X = 1$ and not bother calculating it or using the Poisson approximation. On the other hand, Table E.6 shows binomial probabilities only for particular n from 2 through 20, so that for $n > 20$ the Poisson approximation is often used if p is very small.

To summarize, Figure 5.9 compares the binomial distribution (panel A) and its Poisson approximation (panel B) for the number of defective tires in a sample of 20. The similarities of the two results are clearly evident, thus demonstrating the usefulness of the Poisson approximation even when p is as large as 0.08.

PROBLEMS F OR SECTION 5.6

Learning the Basics

5.74 When should you use the Poisson distribution to approximate the binomial distribution?

5.75 When given the parameters of a binomial distribution, n and p, what are the mean and variance of the Poisson distribution used for approximating the binomial?

5.76 Given a binomial distribution with $n = 100$ and $p = 0.01$, use the Poisson distribution to approximate the following:

a. $P(X = 0)$
b. $P(X = 1)$
c. $P(X = 2)$
d. $P(X \leq 2)$
e. $P(X > 2)$

5.77 Given a binomial distribution with $n = 50$ and $p = 0.004$, use the Poisson distribution to approximate the following:

a. $P(X = 0)$
b. $P(X = 1)$
c. $P(X = 2)$
d. $P(X \leq 2)$
e. $P(X > 2)$

Applying the Concepts

5.78 Based upon past experience, 1% of the telephone bills mailed to households are incorrect. If a sample of 20 bills is selected, find the probability that at least one bill is incorrect. Do this using two probability distributions (the binomial and the Poisson) and briefly compare and explain your results.

5.79 A computer manufacturing company samples incoming computer chips. After receiving a huge shipment of computer chips, the company randomly selects 800 chips. If three or fewer nonconforming chips are found, the entire lot is accepted without inspecting the remaining chips in the lot. If four or more chips are nonconforming, every chip in the entire lot is carefully inspected at the supplier's expense. Assume that the true proportion of nonconforming computer chips being supplied is 0.001. Approximate the probability the lot will be accepted.

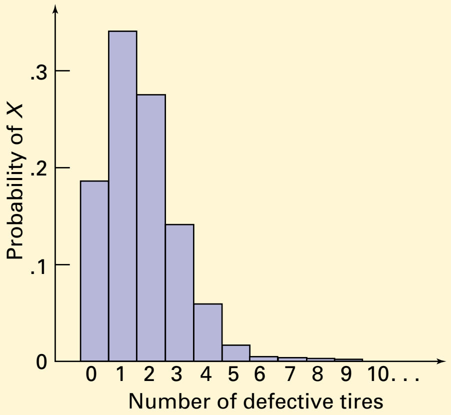
5.80 Last month your company sold 10,000 new watches. Past experience indicates that the probability that a new watch will need repair during its warranty period is 0.002. Approximate the probability that:

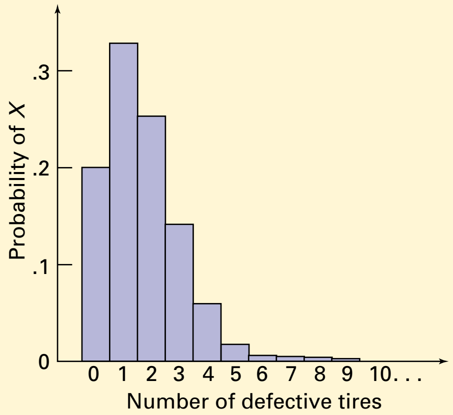
a. zero watches will need warranty work.
b. no more than 5 watches will need warranty work.
c. no more than 10 watches will need warranty work.
d. no more than 20 watches will need warranty work.

FIGURE 5.9

Binomial Distribution and Its Poisson Approximation

Panel A: Binomial Distribution* ($n = 20$ $p = .08$)	Panel B: Poisson Distribution† ($\lambda = np = 1.6$)
$P(X = 0) = .1887$	$P(X = 0) = \frac{e^{-1.6}(1.6)^0}{0!} = .2019$
$P(X = 1) = .3282$	$P(X = 1) = \frac{e^{-1.6}(1.6)^1}{1!} = .3230$
$P(X = 2) = .2711$	$P(X = 2) = \frac{e^{-1.6}(1.6)^2}{2!} = .2584$
$P(X = 3) = .1414$	$P(X = 3) = \frac{e^{-1.6}(1.6)^3}{3!} = .1378$
$P(X = 4) = .0523$	$P(X = 4) = \frac{e^{-1.6}(1.6)^4}{4!} = .0551$
$P(X = 5) = .0145$	$P(X = 5) = \frac{e^{-1.6}(1.6)^5}{5!} = .0176$
$P(X = 6) = .0032$	$P(X = 6) = \frac{e^{-1.6}(1.6)^6}{6!} = .0047$
$P(X = 7) = .0005$	$P(X = 7) = \frac{e^{-1.6}(1.6)^7}{7!} = .0011$
$P(X = 8) = .0001$	$P(X = 8) = \frac{e^{-1.6}(1.6)^8}{8!} = .0002$
$P(X = 9) = .0000$	$P(X = 9) = \frac{e^{-1.6}(1.6)^9}{9!} = .0000$
$P(X = 10) = .0000$	$P(X = 10) = \frac{e^{-1.6}(1.6)^{10}}{10!} = .0000$
$P(X = 20) = \underline{.0000}$	$P(X = 20) = \frac{e^{-1.6}(1.6)^{20}}{20!} = \underline{.0000}$
The probability of discovering two or more defective tires is $1 - [P(0) + P(1)]$ $1 - [.1887 + .3282] = .4831$	The probability of discovering two or more defective tires is approximately $1 - [P(0) + P(1)]$ $1 - [.2019 + .3230] = .4751$





*The binomial probabilities are taken from Table E.6.

†The Poisson probabilities are taken from Table E.7.