ON SKEW QUASI-P-CLASS (Q) OPERATORS, POSIMETRICALLY EQUIVALENT OPERATORS, AND MUTUALLY CLASS (Q) OPERATORS

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Declaration

This thesis is my original work and it has not been submitted elsewhere for a degree in any other university .

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Dedication

To Melville Tatasi and Shanice Ayanja, whose presence and inspiration have been a guiding light throughout this journey .

Abstract

The study of class (Q) operators on Hilbert spaces has been exploited into various classes such as Quasi class (Q) , M-Quasi class (Q) , (n+k)-Class (Q) , Almost class (Q) and (α, β)-class (Q) among others . Results have been proved showing that some of these classes converge to the strong operator topology and results striking relationships between these classes and other general classes were achieved . However , little has been done to expand the results of class (Q) operators into the class of skew-Quasi-p-class (Q) . Hence, in this study, we introduce the category of Skew Quasi-p-class (Q) operators. We examine the fundamental characteristics of this class and establish its connection with other classes, such as quasi-p-normal operators . We also introduce the class of Posimetrically equivalent operators which is a generalization of Metrically equivalent operators , we characterize this class in terms of Complex symmetric operators and study their relations with other equivalence relations such as the class of n-Metrically equivalent operators . We finally introduce the class of Mutually class (Q) operators . Furthermore, we explore the interrelation between this class and other classes in a comprehensive manner . The methodology used include but not limited to , properties of operators like unitary operators , quasi-p-normal operators and skew-adjoint operators . Results shows that the class of skew quasi-p-class (Q) operators have Bishop's property and that they are isoloid and polaroid ; Posimetrically equivalent operators are closed under scalar multiplication and Mutually class (Q) operators are related to class (Q) operators . The study of these classes of (Q) operators will be helpful in the telecommunication industry by generalizing the allocation of network resources basing on priority of network . As a result , high-priority traffic such as video and voice will be given more bandwidth by being transmitted with lower packet loss and delay .

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Chapter 1

Introduction

1.1 Background information

In operator theory, an operator refers to a mathematical object that acts on elements of a given vector space. Specifically, an operator is a mapping that takes vectors from one vector space to another. In the context of operator theory, the vector spaces involved are usually Hilbert spaces, which are complete inner product spaces.

There exist several well-established classes of operators that have been extensively studied and analysed. These classes include but are not limited to self-adjoint operators, unitary operators, normal operators, and compact operators (Furuta, 2001) . Each of these classes possesses unique properties and characteristics that have been extensively explored in the literature.

However, despite the existing classes, there is a need to introduce new classes of operators to further expand the understanding of operator theory and address specific mathematical and practical challenges. The introduction of new classes allows for the exploration of operators with distinct properties and behaviours, providing a more comprehensive framework for analysing and describing various operator characteristics.

By introducing new classes such as skew quasi-p-class (Q) operators, posimetrically equivalent operators, and mutually class (Q) operators, researchers can investigate and understand a broader range of operator properties and relationships. These new classes offer the opportunity to explore novel connections, patterns, and phenomena that may not be captured by the existing operator classes. Moreover, the introduction of new classes paves the way for the development of specialized techniques and tools tailored to the specific properties and behaviours exhibited by operators in these classes. Overall, the introduction of new classes of operators in operator theory enhances the depth and breadth of knowledge in the field and enables researchers to tackle complex problems and advance the understanding of operator behaviour in diverse contexts.

1.2 Basic Concepts

In this section , we outline basic abstractions that will be elemental to our study .

Definition 1.2.1. *In an inner product space, which is defined on a vector space* $\mathcal{V}$ *, there exists a non-negative mapping* $\langle .,. \rangle : \mathcal{V} \text{ x } \mathcal{V} \to \mathbb{K}$ *such that* $\forall\ \S, \dagger \in \mathcal{V}$ *and* $\lambda \in \mathbb{K}$ *; the following axioms hold :*

1 . $\langle \S, \S \rangle \geq 0$ *and* $\langle \S, \S \rangle = 0$ *,if and only if* $\S = 0$*.*

2 . $\langle \S + \dagger, \ddagger \rangle = \langle \S, \ddagger \rangle + \langle \dagger, \ddagger \rangle$

3 . $\langle \lambda \S, \dagger \rangle = \lambda \langle \S, \dagger \rangle$

4 . $\langle \S, \dagger \rangle = \overline{\langle \dagger, \S \rangle}$

$(\mathcal{V}, \langle .,. \rangle)$ *is referred to as an inner product space (Furuta, 2001, Definition 1.1) .*

Definition 1.2.2. *A Hilbert space is a complete inner product space (Kreyszig, 1991) .*

Definition 1.2.3. *An isometry operator in a given Hilbert space* $\mathcal{H}$ *is defined as an operator* $\mathcal{U}$ *that satisfies the following conditions :*

(1) $\|\mathcal{U}\S\| = \|\S\|$ *for every* $\S \in \mathcal{H}$

(1) implies (1')

(1') $(\mathcal{U}\S, \mathcal{U}\dagger) = (\S, \dagger)$, $\forall\ \S, \dagger \in \mathcal{H}$

A unitary operator in a given Hilbert space $\mathcal{H}$ *is an operator* $\mathcal{U}$ *that acts as an isometric, mapping one Hilbert space* $\mathcal{H}$ *onto another Hilbert space* $\mathcal{H}$ *(Furuta, 2001, Section 2.2.1) .*

Definition 1.2.4. *Consider a vector space* $\mathcal{X}$ *over the complex scalars C. If there exists a real number* $\|\S\|$ *for every vector* $\S \in \mathcal{X}$ *that satisfies certain conditions (1), (2), and (3), then* $\|\S\|$ *is defined as the norm of* $\S$ *:*

1 . $\|\S\| \geq 0$ *for all* $\S$ *in* $\mathcal{X}$ *,and* $\|\S\| = 0$ *if and only if* $\S = 0$ *;*

2. $\|\lambda\S\| = |\lambda|\ \|\S\|$ *for all* $\S$ *in* $\mathcal{X}$ *and all complex number* λ *;*

3 . $\|\S + \dagger\| \leq \|\S\| + \|\dagger\|$ *for all* $\S$ *and* $\dagger$ *in* $\mathcal{X}$ *(Kreyszig, 1991) .*

Definition 1.2.5. *A Banach space is a type of normed space that possesses the property of completeness (Bachman & Narici, 2000) .*

Definition 1.2.6. *An operator* $\mathcal{G}$ *in the space of bounded operators* $\mathcal{B}(\mathcal{H})$ *is referred to as:*

(i) Normal when $\mathcal{G}^*\mathcal{G} = \mathcal{G}\mathcal{G}^*$

(ii) Self-adjoint when $\mathcal{G}^* = \mathcal{G}$

(iii) Skew adjoint when $\mathcal{G}^* = -\mathcal{G}$

(iv) An orthogonal projection when $\mathcal{G}^* = \mathcal{G}$ *(idempotent) and* $\mathcal{G}^2 = \mathrm{I}$

(v) Unitary when $\mathcal{G}^*\mathcal{G} = \mathcal{G}\mathcal{G}^* = \mathrm{I}$

(vi) A symmetry when $\mathcal{G} = \mathcal{G}^* = \mathcal{G}^{-1}$

(vii) Isometric when $\mathcal{G}^*\mathcal{G} = \mathcal{I}$

(viii) Partial Isometry when $\mathcal{G} = \mathcal{G}\mathcal{G}^*\mathcal{G}$ *or equivalently when* $\mathcal{G}^*\mathcal{G}$ *is a projection*

(ix) Hyponormal when $\mathcal{G}^*\mathcal{G} \geq \mathcal{G}\mathcal{G}^*$

(x) Quasinormal when $\mathcal{G}(\mathcal{G}^*\mathcal{G}) = (\mathcal{G}^*\mathcal{G})\ \mathcal{G}$

(xi) k-quasinormal when $\mathcal{G}^k(\mathcal{G}^*\mathcal{G}) = (\mathcal{G}^*\mathcal{G})\ \mathcal{G}^k$

(xii) N-normal when $\mathcal{G}^*\mathcal{G}^n = \mathcal{G}^n\mathcal{G}^*$

(xiii) Quasi-Isometry when $\mathcal{G}^{*2}\mathcal{G}^2 = \mathcal{G}^*\mathcal{G}$

(xiv) N-quasinormal when $(\mathcal{G}^*\mathcal{G})\ \mathcal{G}^n = \mathcal{G}^n(\mathcal{G}^*\mathcal{G})$

(xv) Skew-normal when $\mathcal{G}^2 = \mathcal{G}^{*2}$

(xvi) α*-operator when* $\mathcal{G}^3 = \mathcal{G}^*$

(xvii) Q*-operator when* $\mathcal{G}^2\mathcal{G}^{*2} = (\mathcal{G}^*\mathcal{G})^2$ *(Furuta, 2001, Section 2.1.1) .*

Definition 1.2.7. *An operator* $\mathcal{G}$ *in the space of bounded operators* $\mathcal{B}(\mathcal{H})$ *is defined as a* θ*-operator if the commutator [* $\mathcal{G}^* + \mathcal{G}$ *,* $\mathcal{G}^*\mathcal{G}$ *] equals zero (Amjad et al., 2019) .*

Definition 1.2.8. *A bounded linear operator* $\mathcal{G}$ *is considered an* α*-operator if* $\mathcal{G}^3$ *is equal to its adjoint* $\mathcal{G}^*$ *(Jibril, 2010) .*

Definition 1.2.9. *A bounded linear operator* $\mathcal{G}$ *is classified as being in the class (Q) if* $\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{G}^*\mathcal{G})^2$ *(Jibril, 2010) .*

Definition 1.2.10. *A bounded linear operator* $\mathcal{G}$ *is referred to as an Almost Class (Q) operator if* $\mathcal{G}^{*2}\mathcal{G}^2 \geq (\mathcal{G}^*\mathcal{G})^2$ *(Wanjala & Adhiambo, 2021) .*

Definition 1.2.11. *A bounded linear operator $\mathcal{G}$ is considered to be an (n+k)-power (Q) operator when the equation $\mathcal{G}^{2(n+k)}\mathcal{G}^{*2} = (\mathcal{G}^*\mathcal{G})^2$ (Manikandan & Veluchamy, 2018) .*

Definition 1.2.12. *An operator $\mathcal{G}$ in the space of bounded operators $\mathcal{B}(\mathcal{H})$ is referred to as class Q^*-operator when $\mathcal{G}^2\mathcal{G}^{*2} = (\mathcal{G}\mathcal{G}^*)^2$ (Wanjala & Nyongesa, 2021) .*

Definition 1.2.13. *An operator $\mathcal{G}$ in the space of bounded operators $\mathcal{B}(\mathcal{H})$ is referred to as class K^* Quasi-n-class (Q) operator if $(\mathcal{G}^*)^k\mathcal{G}^{*2}\mathcal{G}^{2n} = (\mathcal{G}^*\mathcal{G}^n)^2(\mathcal{G}^*)^k$ (Wanjala & Kiptoo, 2021) .*

Definition 1.2.14. *An operator $\mathcal{G}$ in the space of bounded operators $\mathcal{B}(\mathcal{H})$ is referred to as (α , β)-class (Q) if $\alpha^2\ \mathcal{G}^{*2}\mathcal{G}^2 \leq (\mathcal{G}^*\mathcal{G})^2 \leq \beta^2\mathcal{G}^{*2}\mathcal{G}^2$ for $0 \leq \alpha \leq \beta \leq 1$ (Wanjala & Nyongesa, 2021) .*

Definition 1.2.15. *An operator $\mathcal{G}$ in the space of bounded operators $\mathcal{B}(\mathcal{H})$ is referred to as class (BQ) if $\mathcal{G}^{*2}\mathcal{G}^2$ commutes with $(\mathcal{G}^*\mathcal{G})^2$ (Wanjala & Adhiambo, 2021) .*

Definition 1.2.16. *A bounded linear operator $\mathcal{G}$ is said to belong to Quasi-class (Q) if $\mathcal{G}\mathcal{G}^2\mathcal{G}^{*2}$ = $(\mathcal{G}^*\mathcal{G})^2\mathcal{G}$ (Revathi & Maheswari, 2019) .*

Definition 1.2.17. *A bounded linear operator $\mathcal{G}$ is said to be M quasi-class (Q) operator if $\mathcal{G}\mathcal{G}^2\mathcal{G}^{*2}$ = $\mathrm{M}(\mathcal{G}^*\mathcal{G})^2\mathcal{G}$ for a bounded operator M (Revathi & Maheswari, 2019) .*

Definition 1.2.18. *Two bounded linear operators $\mathcal{S}$ and $\mathcal{G}$ are considered to be metrically equivalent if the equation $\mathcal{S}^*\mathcal{S} = \mathcal{G}^*\mathcal{G}$ holds (Nzimbi et al., 2013) .*

Definition 1.2.19. *Two bounded linear operators $\mathcal{S}$ and $\mathcal{G}$ are regarded as being n-metrically equivalent if the equation $\mathcal{S}^*\mathcal{S}^n = \mathcal{G}^*\mathcal{G}^n$ is satisfied, where n is a positive integer (Wanjala et al., 2020) .*

Definition 1.2.20. *Two bounded linear operators $\mathcal{S}$ and $\mathcal{G}$ are said to be (n,m)-metrically equivalent if $\mathcal{S}^{*m}\mathcal{S}^n = \mathcal{G}^{*m}\mathcal{G}^n$ for positive integers n and m (Wanjala & Nyongesa, 2021) .*

Definition 1.2.21. *If there exists an invertible operator $\mathcal{N}$ such that $\mathcal{S}^*\mathcal{S}$ can be expressed as $\mathcal{N}^{-1}(\mathcal{G}^*\mathcal{G})\mathcal{N}$ and $\mathcal{S}^* + \mathcal{S}$ can be expressed as $\mathcal{N}^{-1}(\mathcal{G}^* + \mathcal{G})\mathcal{N}$, then the bounded linear operators $\mathcal{S}$ and $\mathcal{G}$ are regarded as almost similarly equivalent (Musundi et al., 2013) .*

Definition 1.2.22. *If $\mathcal{G}$ is a bounded linear operator, it is considered isoloid if all isolated points of the spectrum of $\mathcal{G}$ i.e $\sigma(\mathcal{G})$; are also elements of the point spectrum $\sigma_p(\mathcal{G})$. $\mathcal{G}$ is polaroid if every isolated point of $\sigma(\mathcal{G})$ is a pole resolvent of $\mathcal{G}$ (Muneo et al., 2018) .*

Definition 1.2.23. *A bounded linear operator $\mathcal{G}$ is classified as n-perinormal if the inequality $\mathcal{G}^n\mathcal{G}^{*n} \geq (\mathcal{G}^*\mathcal{G})^n$ holds true for a positive integer n (Hongliang & Fei, 2014) .*

Definition 1.2.24. *Two bounded linear operators $\mathcal{S}$ and $\mathcal{G}$ are said to be mutually normal if $\mathcal{G}\mathcal{G}^* = \mathcal{S}^*\mathcal{S}$ and $\mathcal{G}^*\mathcal{G} = \mathcal{S}\mathcal{S}^*$ (Jibril, 1999) .*

Definition 1.2.25. *The operator $\mathcal{G}$ belonging to the bounded operators on Hilbert space $\mathcal{H}$ is referred to as* A*-normal if the equation $\mathcal{G}^\sharp\mathcal{G} = \mathcal{G}\mathcal{G}^\sharp$ is satisfied (Adel, 2012) .*

Definition 1.2.26. *The bounded linear operator $\mathcal{G}$ in the space of bounded operators on Hilbert space $\mathcal{H}$ is considered quasi-p-normal if the operators $(\mathcal{G} + \mathcal{G}^*)$ and $(\mathcal{G}^*\mathcal{G})$ commute (Senthilkumar & Revathi, 2019) .*

Definition 1.2.27. *The bounded linear operator $\mathcal{G}$ in the space of bounded operators on Hilbert space $\mathcal{H}$ is defined as* A*-quasi normal if the equation $\mathcal{G}(\mathcal{G}^\sharp\mathcal{G}) = (\mathcal{G}\mathcal{G}^\sharp)\mathcal{G}$ holds (Panayappan & Sivamani, 2012).*

Definition 1.2.28. *The bounded linear operator $\mathcal{G}$ in the space of bounded operators on Hilbert space $\mathcal{H}$ is considered to be (n, m)-normal if the equation $\mathcal{G}^{*m}\mathcal{G}^{n} = \mathcal{G}^{n}\mathcal{G}^{*m}$ holds true, where n and m are positive integers (Eiman & Mustafa, 2016) .*

Definition 1.2.29. *For a bounded operator $\mathcal{G}$ in the space of bounded operators on Hilbert space $\mathcal{H}$, it is said to have Bishop's property if, for all sequences of an analytic function $f_{\S} : U \longrightarrow H$, where U is an open subset of the complex plane, the expression $(\lambda - \mathcal{G})f_{\S}\lambda$ approaches zero as $\S$ tends to infinity uniformly on all compact subsets of U, and $f_{\S}\lambda$ tends to zero as $\S$ tends to infinity locally uniformly on U (Ould, 2014) .*

Definition 1.2.30. *The orthogonal complement of a closed subspace $\mathcal{M}$ in a Hilbert space $\mathcal{H}$ is defined as the subspace $\mathcal{M}^{\perp}$ which consists of all vectors in $\mathcal{H}$ that are orthogonal to every vector in $\mathcal{M}$,that is $\mathcal{M}^{\perp} = \{\mathrm{z} \in \mathcal{H} : (z, x) = 0 \ for\ all\ x \in \mathcal{M}\}$ (Furuta, 2001, Section 1.3) .*

1.2.1 Statement of the problem

Operators in Class (Q) and metrically equivalent operators are crucial in the telecommunications industry because they use Quality of Service (QoS) mechanisms to ensure customers receive specific service levels. These operators offer different Service Level Agreements (SLAs) based on customer requirements. For example, customers needing high availability and low delays receive higher SLAs compared to those with less stringent needs.

However, scalability is a significant challenge in this field. As network complexity and size increase with higher traffic volumes and data, maintaining these service levels becomes difficult. Our study addresses this issue by generalizing Class (Q) and metrically equivalent operators into skew quasi-p-class (Q), Mutually class (Q), and posimetrically equivalent operators. This generalization allows for handling more data, enabling the creation of more flexible and reusable codes using supercomputers, thus reducing laborious processes.

1.2.2 General objective of the study

The general objective of the study was to introduce new classes of operators , that is , skew Quasi-p-class(Q) operators , Posimetrically equivalent operators and Mutually class (Q) operators .

1.2.3 Specific objectives of the study

The specific objectives of the study are to :

1 . Analyse algebraic properties of skew quasi-p-class (Q) operators;

2 . Determine the relation of the class of posimetrically equivalent operators with other equivalence relations;

3 . Establish the relation of Mutually class (Q) operators with other classes.

1.2.4 Significance of the study

The study of skew quasi-p-class (Q), posimetrically equivalent, and mutually class (Q) operators could significantly impact the telecommunications industry. By leveraging their properties, it may be possible to develop codes using supercomputers to manage network traffic flow. This can be achieved by applying various Quality of Service (QoS) parameters within a broader framework, as these operators generalize class (Q) operators.

QoS parameters are essential for evaluating the performance of different types of network traffic. They include metrics such as packet loss during data transmission and bandwidth, which measures the amount of data that can be transmitted over a network in a given time. Utilizing these parameters can ensure that traffic is routed along the most efficient paths, thereby preventing congestion and enhancing overall network performance.

Chapter 2

Literature Review

The substantial growth in the investigation of operators within the traditional Hilbert space $\mathcal{H}$ has resulted in the extensive expansion of the class of normal operators, leading to the emergence of numerous subclasses. These subclasses, such as n-normal, perinormal , mutually-normal, quasi-normal, quasi-p-normal, skew-normal, skew quasi-p-normal, and (n, m)-normal , have been developed to capture specific properties exhibited by operators.

In the context of n-power normal operators, Jibril (2008) focused on this class and established connections between 3-normal and 2-normal operators. It was demonstrated that if an operator, denoted as $\mathcal{G}$, is both one-to-one and satisfies the conditions of being 2-normal and 3-normal, then it is a normal operator. Alzuraiqi and Patel (2016) delved into the realm of n-normal operators and showed that this class is not essentially normal or hyponormal, uncovering intriguing properties specific to this subclass.

Continuing the exploration of n-normal operators, Muneo and Biljana (2018) analysed the spectral picture associated with this class. Their research outcomes demonstrated that the point spectrum and approximate point spectrum of n-normal operators coincide, providing insights into the spectral characteristics within this specific subclass.

Furthermore, Eiman and Mustafa (2016) investigated the class of (n,m)-normal operators, further expanding the understanding of operators in the Hilbert space $\mathcal{H}$. By linking these two paragraphs, it becomes evident that the study of operators has progressed through the examination of various subclasses, each offering unique insights into the behaviour and properties of operators in $\mathcal{H}$.

Continuing with the generalization of normal operators, a further extension was made to the class of (n,m)-normal operators. $\mathcal{G}$ is considered (n,m)-normal whenever $\mathcal{G}^{*m}\mathcal{G}^{n} = \mathcal{G}^{n}\mathcal{G}^{*m}$ for all $0 \leq m, n$. Eiman and Mustafa (2016) showed that this category exhibits a correlation with the class of (n,m)-quasinormal operators, unveiling an intriguing association between these subclasses .

Another significant contribution to the study of operators came from Mahmoud (2016) , who introduced the concept of square-normal operators. He provided an example that distinguished this class from the class of normal operators, highlighting their distinct properties. Additionally, Mahmoud (2016) gave a condition for a normal operator to be classified as a square-normal operator.

Veluchamy and Manikandan (2016) conducted a study on n-power quasi-normal operators . It was observed that an operator $\mathcal{G}$ is considered n-power quasi-normal whenever it is self-adjoint. Furthermore, it was expanded upon by establishing that the adjoint of an operator $\mathcal{G}$ is an n-power quasinormal operator if and only if it satisfies both the conditions of being n-power quasi-normal and self-adjoint. Taking the study of normal operators into deformed aspects, Schoichi (2002) introduced the concept of q-deformed normal operators. If a densely defined operator $\mathcal{G}$ on a Hilbert space $\mathcal{H}$ satisfies the equation $q\mathcal{G}^{*}\mathcal{G} = \mathcal{G}\mathcal{G}^{*}$, where q is a deformation parameter and q is not equal to 1, it is classified as a q-deformed operator . Results linking this class with other classes, such as q-quasinormal operators, were also discussed, expanding the understanding of deformed operator structures.

Expanding the analysis to the semi-Hilbertian space, Adel (2012) introduced the study of normal operators in this context. An operator $\mathcal{G} \in \mathcal{B}_{\mathcal{A}}(\mathcal{H})$ is labelled A-normal if $[\mathcal{G}^{\sharp}, \mathcal{G}] = 0$, where $\mathcal{G}^{\sharp}$ represents the adjoint of $\mathcal{G}$. Notably, an A-normal operator is normal when $\mathcal{G}^{\sharp} = \mathcal{G}^{*}$. Adel (2012) provided results concerning the inequality of this class, further advancing the understanding of A-normal operators. The exploration of A-normal operators was extended by Panayappan and Sivamani (2012) to A-quasinormal operators. In this case, an operator $\mathcal{G} \in \mathcal{B}_{\mathcal{A}}(\mathcal{H})$ is considered A-quasinormal if $\mathcal{G}(\mathcal{G}^{\sharp}\mathcal{G}) = (\mathcal{G}^{\sharp}\mathcal{G})\mathcal{G}$. Meenambika et al. (2018) delved into the study of skew-normal , defining a bounded $\mathcal{G}$ as skew normal whenever $(\mathcal{G}\mathcal{G}^{*})\mathcal{G} = \mathcal{G}(\mathcal{G}^{*}\mathcal{G})$. The characteristics of this category were explored, and a finding concerning the connection between skew-normal operators and self-adjoint operators was presented.

Shifting focus to k-quasi-normal operators, Senthilkumar et al. (2012) conducted this study. An operator $\mathcal{G}$ is said to be k-quasi-normal if $\mathcal{G}(\mathcal{G}^{*}\mathcal{G})^{k} = (\mathcal{G}^{*}\mathcal{G})^{k}\mathcal{G}$, where the multiplication is prompted by Radon-Nikodym derivative with $\lambda(\mathcal{G}^{k})^{-1}$ being the measure with regard to λ . This class incorporated the concept of composition, composite multiplication, and weighted composition operators.

Senthilkumar and Revathi (2019) further contributed to the field by studying Quasi-p-normal operators and Quasi-n-p-normal operators. They characterized these classes in terms of composition, composite multiplication, and weighted composition operators. A bounded linear operator $\mathcal{G}$ is considered Quasi-p-normal if $(\mathcal{G} + \mathcal{G}^{*})$ commutes with $\mathcal{G}^{*}\mathcal{G}$.

The investigation of the spectrum of n-perinormal operators raised a concern regarding the equality of the joint approximate and the approximate spectrum of an n-perinormal (Mecheri & Braha, 2012) . This concern was addressed by Hongliang and Fei (2014) who noted that indeed the joint approximate and approximate spectrum of n-perinormal operators are equal. Hongliang and Fei (2014) also covered the tensor products of the class of n-perinormal operators.

Further advancements were made by Wanjala and Adhiambo (2021) where they introduced Furuta inequalities to the class of n-perinormal operators. They achieved this by introducing the class of 2n-perinormal operators, where $\mathcal{G}^{*2n}\mathcal{G}^{2n} \geq (\mathcal{G}^*\mathcal{G})^{2n}$ for all positive integers n and a bounded $\mathcal{G}$. This class was characterized in terms of isometric operators.

The exploration of the class of normal operators has led to the expansion into other related classes such as hyponormal , square hyponormal, log hyponormal , and M-hyponormal operators , among others. These classes relax the conditions for normality and provide a broader understanding of operator properties.

Arora and Ramesh (1980) addressed some results related to M-hyponormal operators. It was demonstrated that every isolated point within the spectrum of this class corresponds to an eigenvalue of the operator. Additionally, the Weyl spectrum of $\mathcal{G}$ was found to be the same as the similarity between the spectrum of $\mathcal{G}$ and its isolated points. Furthermore, a connection between the class of M-hyponormal operators together with normal were established, demonstrating that if a bounded operator $\mathcal{G}$ has a single limit point in its spectrum, it tends to be normal.

Ilyas and Reyaz (2012) studied classes related to p-hyponormal operators. An operator $\mathcal{G}$ belonging to the bounded operators on Hilbert space $\mathcal{H}$ is classified as p-hyponormal whenever the inequality $(\mathcal{G}^*\mathcal{G})^p \geq (\mathcal{G}\mathcal{G}^*)^p$ holds true for non-negative p. They linked this class to $^*p$-

paranormal operators and monotonicity of ${}^*\mathcal{A}(p,q)$ operators. The class ${}^*\mathcal{A}(p,q)$ is defined as $|\mathcal{G}|^{2q} \geq (|\mathcal{G}|^q|\mathcal{G}^*|^{2p}|\mathcal{G}|^q)^{\frac{q}{p+q}}$ for non-negative p and q.

Muneo et al. (2019) introduced square hyponormal operators and investigated their spectral properties. An operator $\mathcal{G}$ is said to be square hyponormal if $(\mathcal{G}^*\mathcal{G})^2 \geq (\mathcal{G}\mathcal{G}^*)^2$. It was shown that this class satisfies the single value extension property, which means that it has unique analytic solutions for certain equations. An operator $\mathcal{G}$ possess a single valued extension property whenever every neighborhood U of x_o at $x_o \in \mathbb{C}$, is the only analytic solution f for the equation $(\mathcal{G} - x)f(x) = 0$ for every $x \in U$ is the constant function $f \equiv 0$ (Ould, 2014) .

Ould (2014) presented fascinating results touching on n-power quasi-normal and n-power k-quasinormal operators. In particular, it was demonstrated that these classes possess single value extension property thus exhibiting Bishop's property. The single value extension property ensures that unique analytic solutions exist for specific equations involving these operators. Furthermore, Ould (2014) proved that the category of n-power k-quasi-normal is preserved under unitary equivalence and scalar multiplication.

It is evident that the investigation of operators, whether in the conventional Hilbert space or the semi-Hilbertian space, continues to expand and be characterized in diverse ways. The class of (Q) operators, in particular, has garnered significant attention among researchers. The study of (Q) operators has been intensified, diversified, and explored extensively. The objective of this study is to further extend the examination of this particular class of operators.

Jibril (2010) introduced the class of (Q) operators and extensively investigated their intriguing fundamental properties. The study by Jibril (2010) revealed notable characteristics of this class.

For example, it was demonstrated that if $\mathcal{G}$ belongs to class (Q), then its adjoint is also a class (Q) operator. Additionally, Jibril (2010) established that class (Q) operators converge to the strong operator topology. Jibril (2010) also investigated several interrelationships between the class of (Q) operators and other operator categories, such as quasinormal , θ-operators, isometry and normal operators.

The class (Q) was characterized in terms of isometry where interesting properties were covered (Jibril, 2010). Additionally, a counterexample was provided to illustrate that the converse does not hold universally. Jibril (2010) also established a connection between this class and the class of normal operators, demonstrating that if an operator $\mathcal{G}^2$ is both normal and belongs to class (Q), then it is a normal operator. Furthermore, it was demonstrated that this class does not preserve similarity.

Paramesh et al. (2019) expanded the notion of class (Q) operators by introducing the concept of n-power class (Q) operators. The paper delved into the fundamental properties of this class and presented a result demonstrating that it is not generally a normal operator (*see example 3.4*). Paramesh et al. (2019) also established a correlation between this class and the category of n-normal operators.

As per *Theorem 3.3* by Paramesh et al. (2019), if an operator $\mathcal{G} \in \mathcal{B}(\mathcal{H})$ is categorized as n-normal, it is also considered an n-power class (Q) operator. Moreover, according to *Theorem 3.6* , if $\mathcal{G}$ is n-power class (Q) operator and $\mathcal{G}$ is quasi-n-normal, it can be categorized as an n+1 power class (Q) .

Manikandan and Veluchamy (2018) made a significant contribution by introducing (n+k) power class (Q), with n being a positive definite integer and $0 \leq k$. This new class was accompanied by the characterization of new theorems. Notably, it was shown that if a bounded operator $\mathcal{G}$ is (n+k)-normal, then it falls into the category of (n+k)-power class (Q) operators. Furthermore, Manikandan and Veluchamy (2018) further advanced the results on (n+k) power class (Q) class . Manikandan and Veluchamy (2018) characterized class (Q) operators in terms of complex symmetric operators. As per Manikandan and Veluchamy (2018), *Theorem 2.13* , whenever $\mathcal{G}$ is class (Q) in addition to being complex symmetric operator, then the equation $\mathcal{G}^2\mathcal{G}^{*2} = (\mathcal{G}\mathcal{G}^*)^2$ holds true.

Revathi and Maheswari (2019) introduced a new class called Quasi-class (Q) operators, which builds upon the class (Q) operators. The paper investigated the basic properties of this class and established interesting connections between Quasi-class (Q) operators and self-adjoint operators.

According to Revathi and Maheswari (2019), *Theorem 2.7* , if $\mathcal{G}$ is a self-adjoint operator that is quasi-class (Q) with the existence of $\mathcal{G}^{-1}$, then $\mathcal{G}^{-1}$ is similarly quasi-class (Q). *Theorem 2.8* by Revathi and Maheswari (2019) states that whenever $\mathcal{G}$ is a quasi-class (Q) with $\mathcal{S}$ being self-adjoint that commutes with $\mathcal{G}$, then $\mathcal{S}\mathcal{G}$ is quasi-class (Q) .

Furthermore, Revathi and Maheswari (2019) in *Theorem 2.11* established that for a self-adjoint operator $\mathcal{G}$ and any operator $\mathcal{S}$ on $\mathcal{H}$, the operator $\mathcal{S}^*\mathcal{G}\mathcal{S}$ is a quasi-class (Q) operator. These results highlight the relationship between quasi-class (Q) operators and self-adjoint operators. The paper also established connections between quasi-class (Q) operators and the classes of quasi-normal operators and isometries. According to *Theorem 2.9* by Revathi and

Maheswari (2019) , if $\mathcal{G} \in \mathcal{B}(\mathcal{H})$ is quasinormal, then $\mathcal{G}$ is classified as a quasi-class (Q) operator. Similarly, *Theorem 2.9* and *Theorem 2.10* by Revathi and Maheswari (2019) states that if $\mathcal{G} \in \mathcal{B}(\mathcal{H})$ is an isometry, then $\mathcal{G}$ is also a quasi-class (Q) operator. These findings demonstrate the relationship between quasi-class (Q) operators and other operator classes.

Later on Revathi and Maheswari (2019) extended quasi class (Q) into M-quasi class (Q) where M is bounded on $\mathcal{H}$. Similarly basic properties of this class were investigated . In particular , results showed that the sum of two M quasi class (Q) and the product of two M quasi class (Q) is still M quasi class (Q).

In their research , Wanjala and Nyongesa (2021) extended the study of class (Q) operators to a new class called (α, β)-class (Q), where $0 \leq \alpha \leq \beta \leq 1$. Wanjala and Nyongesa (2021) explored several interesting algebraic properties of this class and made notable discoveries. One such finding revealed that if an operator $\mathcal{G}$ is classified as (α, β)-class (Q), then its adjoint, $\mathcal{G}^*$, also falls into the (α, β)-class (Q). The (α, β)-class (Q) was further connected to other operator classes, including the (α, β)-normal operator . Additionally, the researchers established a link between the (α, β)-class (Q) and unitary operators, presenting a characterization of the (α, β)-class (Q) in terms of unitary operators. In their work , Wanjala and Adhiambo (2021) observed that by relaxing the conditions for the class (Q) operators, it coincides with the class of (M, n) operators, also known as almost class (Q) operators, when the parameter n is equivalent to two .

Building upon the concept of almost class (Q) operators, Wanjala and Adhiambo (2021) further generalized it to (n, m)-almost class (Q) . This generalization considered positive integers n and m, and the paper provided several results regarding this class. The remarkable discovery was made that when an operator $\mathcal{G}$ is labeled as (n, m)-almost class (Q) and there is a unitary operator $\mathcal{S}$ that is equivalent to $\mathcal{G}$, then $\mathcal{S}$ also falls into the category of (n, m)-almost class (Q). Additionally, Wanjala and Adhiambo (2021) presented some results related to (M, n) and n-power hyponormal. These findings shed light on the properties and relationships of these operator classes.

The concept of k^*-Quasi-n-class (Q) operators was introduced by Wanjala and Kiptoo (2021) as an extension of class (Q) operators. The paper discussed the basic properties of this class and various findings were presented.

Later on ,Wanjala and Adhiambo (2021) introduced the class of (BQ) operators . An operator $\mathcal{G}$ is classified as (BQ) if the commutator of $\mathcal{G}^{*2}\mathcal{G}^2$ with $\mathcal{G}^*\mathcal{G}$ exists. The findings presented by Wanjala and Adhiambo (2021) provided evidence that any operator which is unitarily equivalent to a (BQ) operator is likewise categorized as (BQ). Additionally, they provided a result showing that any operator belonging to the class (Q) is also in the (BQ) class. Wanjala and Adhiambo (2021) extended the class of (BQ) operators to (nBQ) operators, considering positive integer values of n. They made an intriguing discovery by establishing a connection between the (nBQ) class and n-power class (Q) operators. Specifically, they demonstrated that if $\mathcal{G}$ is a complex symmetric operator for which the commutator $\mathcal{C}$ commutes with $(\mathcal{G}^*\mathcal{G})^2$, it implies $\mathcal{G}$ is n-power class (Q) .

In the work by Wanjala and Nyongesa (2021) , a new class called (Q^*) operators was introduced. A result by Wanjala and Nyongesa (2021) provided an example that illustrated (Q^*)

class is distinct from the class (Q). Furthermore, Wanjala and Nyongesa (2021) presented a significant result that established a connection between the (Q^*) class and square-hyponormal operators. Through the characterization of (α , β)-class (Q) operators, there was demonstratation for the relationship between the (Q^*) class and square-hyponormal operators.

In the work presented by Jibril (1999) , a new class called mutually normal operators was introduced and thoroughly investigated. Results explored the intriguing properties of this class and established connections between mutually normal operators and other operator classes.

Of particular interest was a result that demonstrated a relationship between mutually normal and hyponormal operators. The paper provided insights into this connection, shedding light on the interplay between these two classes of operators.

It is evident that there has been no prior research conducted on the class of skew-quasi-p-class (Q) and mutually class (Q) operators. Therefore, this study aims to introduce and investigate these two classes, namely skew-quasi-p-class (Q) and mutually class (Q) operators, by examining their fundamental properties.

Furthermore, the study aims to establish connections between these two classes and other general classes of operators. By exploring the relationships and inter-dependencies with other operator classes, this research intends to provide a comprehensive understanding of the characteristics and behavior of skew-quasi-p-class (Q) and mutually class (Q) operators.

The exploration of equivalent classes of operators has been a subject of study by numerous authors. One such class is that of almost similarity . While studying almost similarity of operators , Jibril (1996) provided intriguing results that characterized the class of almost similar operators in relation to normal operators.

Building upon the work done by Jibril (1996) , Musundi et al. (2013) further investigated the class of almost similar operators and established that it possesses the property of being an equivalence relation. This finding by Musundi et al. (2013) demonstrated the relationship and equivalence among operators within the class of almost similar operators.

In his work, Sadoon (1996) focused on the concept of nearly equivalent operators. He defined $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{G} \in \mathcal{B}(\mathcal{H})$ being nearly equivalent if their respective self-adjoint products, $\mathcal{S}^*\mathcal{S}$ and $\mathcal{G}^*\mathcal{G}$, are similar. Furthermore , Sadoon (1996) investigated the class of nearly normal operators. He provided an example that demonstrated that nearly normal operators are not necessarily normal. Additionally, he presented a condition under which the notions of near normality and normality coincide, suggesting that under certain conditions, these two properties are considered equivalent for operators .

Nzimbi et al. (2013) conducted a study on the metric equivalence of operators. The paper presented results that established connections between metric equivalence and other general classes of operators. Notably, the class of metrically equivalent operators was found to be linked to the classes of quasinormal and normal operators. Furthermore, Nzimbi et al. (2013) provided several results that showcased the relationship between the class of metrically equivalent operators and other equivalence relations, including unitary equivalence. These findings contributed to a better understanding of the interplay between metric equivalence and other forms of equivalence among operators.

In a subsequent work , Wanjala et al. (2020) extended the study of metrically equivalent operators to n-metric equivalent operators. The paper examined the properties of this class and established its relationship with general classes such as quasinormal and k-quasinormal operators. One notable result presented by Wanjala et al. (2020) addressed the condition under which n-metric equivalence of operators becomes equivalent to metric equivalence, specifically when the value of n is equal to two. This result provided insights into the connection between n-metric equivalence and metric equivalence in certain cases. Furthermore, Wanjala et al. (2020) established a link between n-metric equivalence and n-normal operators. This connection shed light on the relationship between these two operator classes.

In a later study by Wanjala and Nyongesa (2021) , n-metrically equivalent concept was extended and generalized to (n,m)-metrically equivalent class. This development led to the discovery of several interesting findings, establishing connections between (n,m)-metrically equivalent class and other important classes, including quasi-isometries and (n,m)-class (Q) operators.

In a subsequent study by Wanjala and Adhiambo (2021) , the concept of n-metric equivalence of operators was extended to the semi-Hilbertian space. This class of operators was thoroughly investigated, and various properties were examined. Additionally, significant connections between n-metric and the A-normal operators were established and analyzed. The findings from this research shed light on the relationship between these two important classes of operators in the semi-Hilbertian space.

From this literature review, it is evident that, just like the classes of skew-quasi-p-class (Q) and Mutually class (Q) operators, no research has been conducted on the intriguing class of posimetrically equivalent operators. Thus, the primary objective of this study is to introduce and investigate the class of posimetrically equivalent operators, unveiling their inherent characteristics and establishing their connections to various general classes and equivalent classes. This research aims to fill the existing gap in the literature and contribute to a deeper understanding of this unexplored class of operators.

Chapter 3

Research Methodology

3.1 Introduction

In pursuit of our specific objectives, we employed various methodologies, including the utilization of properties of normal operators, n-normal operators, isometries, unitary operators, adjoint operators related operators, as well as classes related to metrically equivalent operators like n-metrically equivalent operators. Throughout the study, we examined the essential properties of unitary operators and adjoint operators. To establish these properties, we relied on well-known Theorems, Propositions, and Corollaries which we restate here .

3.2 Fundamental principles

The following well known results were useful in our results .

Theorem 3.2.1. *Suppose $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ is an n-normal operator. In that case, $\mathcal{S}$ is both isoloid and polaroid (Muneo et al., 2018) .*

Theorem 3.2.2. *If $\mathcal{S}$ is an n-power normal operator, it satisfies Bishop's property (Stella & Vijayalakshmi, 2015 , Theorem 2.5) .*

Theorem 3.2.3. *If $\mathcal{T}$ is an operator on $\mathcal{H}$, it follows that $\mathcal{T}^*$ is equally an operator on the Hilbert space $\mathcal{H}$ and the following are true :*

(a) $\|\mathcal{T}^*\| = \|\mathcal{T}\|$.

(b) $(\mathcal{T}_1 + \mathcal{T}_2)^* = \mathcal{T}_1^* + \mathcal{T}_2^*$.

(c) $(\alpha\mathcal{T})^* = \overline{\alpha}\,\mathcal{T}^*$ *for every* $\alpha \in \mathbb{C}$ *.*

(d) $(\mathcal{T}^*)^* = \mathcal{T}$ *.*

(e) $(\mathcal{S}\mathcal{T})^* = \mathcal{T}^*\mathcal{S}^*$ *(Furuta, 2001, Section 2.2.1) .*

3.2.1 n-metric equivalence of operators

In this section , we restate some results of n-metrically equivalent operators as covered by Wanjala et al. (2020) that are useful in our findings .

Theorem 3.2.4. *If* $\mathcal{S}$ *is an n-normal operator and* $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ *is unitarily equivalent to* $\mathcal{S}$*, then* $\mathcal{T}$ *is an n-normal (Wanjala et al., 2020, Theorem 2.1) .*

Corollary 3.2.5. *An operator* $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ *is an n-normal if and only if* $\mathcal{T}$ *and* $\mathcal{T}^*$ *are n-metrically equivalent (Wanjala et al., 2020, Corollary 2.2) .*

Theorem 3.2.6. *If* $\mathcal{S}$ *and* $\mathcal{T}$ *are unitarily 2-metrically equivalent operators and* $\mathcal{S}$ *is quasinormal , then* $\mathcal{T}$ *is quasinormal (Wanjala et al., 2020, Theorem 3.2) .*

Theorem 3.2.7. *If* $\mathcal{S}$ *and* $\mathcal{T}$ *are unitarily 2-metrically equivalent operators then they are metrically equivalent provided they are idempotent (Wanjala et al., 2020, Theorem 3.3) .*

3.2.2 n,m-metrically equivalent operators.

In this section , we restate some results of (n,m)-metrically equivalent operators as covered by Wanjala and Nyongesa (2021) that are useful in our findings .

Theorem 3.2.8. *If* $\mathcal{S}$ *is an (n,m)-normal operator and* $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ *is unitarily equivalent to* $\mathcal{S}$ *,then* $\mathcal{T}$ *is an (n,m)-normal (Wanjala & Nyongesa, 2021 , Theorem 2.1) .*

Corollary 3.2.9. *An operator $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ is (n,m)-normal if and only if $\mathcal{T}$ and $\mathcal{T}^*$ are (n,m)-metrically equivalent (Wanjala & Nyongesa, 2021 , Corollary 2.2) .*

Chapter 4

Skew Quasi-P-Class (Q) Operator

4.1 Introduction

In this chapter, we focus on our first objective, which is the study of Skew-Quasi-P-Class (Q). We delve into the properties of this class and explore its connections and relationships with other operator classes such as quasi-p-normal and (n,m)-normal. Throughout the chapter, we provide a comprehensive analysis of the Skew-Quasi-P-Class (Q), shedding light on its unique characteristics and its interplay with other classes in the field of operator theory.

Definition 4.1.1. *$\mathcal{T} \in \mathcal{B}(\mathcal{H})$ is skew quasi-p-class (Q) whenever $(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$. We shall denote this class as $[V]$.*

Theorem 4.1.1. *Let $\mathcal{T} \in [V]$, then so are any ;*

1. *$\psi\mathcal{T}$ for any $\psi \in \mathbb{R}$.*
2. *Every $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ unitarily equivalent to $\mathcal{T}$.*

Proof.

(i). Let $\mathcal{T} \in [V]$, then ;

$$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T}+\mathcal{T}^*) = (\mathcal{T}+\mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2 \tag{4.1}$$

$$= ((\psi\mathcal{T})^{*2}(\psi\mathcal{T})^2)(\psi\mathcal{T}+(\psi\mathcal{T})^*) \tag{4.2}$$

$$= \overline{\psi}^2\mathcal{T}^{*2}\psi^2\mathcal{T}^2(\psi\mathcal{T}+\overline{\psi}\mathcal{T}^*) \tag{4.3}$$

$$= \psi^5(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T}+\mathcal{T}^*) \tag{4.4}$$

and;

$$= (\psi\mathcal{T}+(\psi\mathcal{T})^*)((\psi\mathcal{T})^*\psi\mathcal{T})^2 \tag{4.5}$$

$$= (\psi\mathcal{T}+\overline{\psi}\mathcal{T}^*)\psi^2\overline{\psi}^2(\mathcal{T}^*\mathcal{T})^2 \tag{4.6}$$

$$= \psi^5(\mathcal{T}+\mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2. \tag{4.7}$$

hence from 4.4 and 4.7 $\psi\mathcal{T}$ is skew quasi-p-class (Q) .

(ii). Suppose $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ is unitarily equivalent to $\mathcal{T}$, $\exists$ unitary operator $\mathcal{U} \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{T} = \mathcal{U}^*\mathcal{S}\mathcal{U}$. Then ;

$$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T}+\mathcal{T}^*) = (\mathcal{U}^*\mathcal{S}\mathcal{U}+\mathcal{U}^*\mathcal{S}^*\mathcal{U})(\mathcal{U}^*\mathcal{S}^*\mathcal{U}\mathcal{U}^*\mathcal{S}^*\mathcal{U}\mathcal{U}^*\mathcal{S}\mathcal{U}\mathcal{U}^*\mathcal{S}\mathcal{U}) \tag{4.8}$$

$$= (\mathcal{U}^*\mathcal{S}\mathcal{U}+\mathcal{U}^*\mathcal{S}^*\mathcal{U})(\mathcal{U}^*\mathcal{S}^{*2}\mathcal{S}^2\mathcal{U}) \tag{4.9}$$

$$= (\mathcal{T}\mathcal{U}^*\mathcal{U}+\mathcal{T}^*\mathcal{U}^*\mathcal{U})(\mathcal{T}^{*2}\mathcal{U}^*\mathcal{U}\mathcal{T}^2) \tag{4.10}$$

$$= (\mathcal{T}+\mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2. \tag{4.11}$$

Since $(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T}+\mathcal{T}^*) = (\mathcal{T}+\mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$

then;

$(\mathcal{S}^{*2}\mathcal{S}^2)(\mathcal{S}+\mathcal{S}^*) = (\mathcal{S}+\mathcal{S}^*)(\mathcal{S}^*\mathcal{S})^2.$ □

Theorem 4.1.2. *Let $\mathcal{T}$ be a self-adjoint , then $\mathcal{T} \in [V]$.*

Proof. Suppose $\mathcal{T} \in [V]$;

$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$

$$= (\mathcal{T}^2\mathcal{T}^2)(\mathcal{T} + \mathcal{T}) = 2\mathcal{T}^5 \tag{4.12}$$

$$= (\mathcal{T} + \mathcal{T})(\mathcal{T}\mathcal{T})^2 = 2\mathcal{T}^5 \tag{4.13}$$

From 4.12 and 4.13 ; $\mathcal{T}$ is skew quasi-p-class (Q) operator.

□

Remark 4.1.1. *The counterexample below shows that the class of skew quasi-p-class (Q) does not preserve operator similarity. This highlights a limitation in the preservation of similarity within the class, emphasizing the importance of considering the specific properties of operators within the skew quasi-p-class (Q).*

Example 4.1.1. *Let $\mathcal{T}$ be an operator acting on $\mathbb{R}^2$ such that $\mathcal{T} = \begin{bmatrix} e^{In(2)} & e^{i\pi}+1 \\ e^{i\pi}+1 & e^{2In(2)} \end{bmatrix}$ and $\mathcal{X} = \begin{bmatrix} e^{2In(2)} & e^{In(2)} \\ e^{In(2)} & e^{In(2)} \end{bmatrix}$. It is easy to verify that ;*

$(\mathcal{T}^{*2}\mathcal{T}^2)\,(\mathcal{T}+\mathcal{T}^*) = \begin{bmatrix} e^{In(64)} & e^{i\pi}+1 \\ e^{i\pi}+1 & e^{In(2048)} \end{bmatrix} = (\mathcal{T}+\mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$ *hence is skew quasi-p-class (Q). Now let ;*

$\mathcal{X}\mathcal{T}\mathcal{X}^{-1} = \begin{bmatrix} e^{i\pi}+1 & e^{In(16)} \\ e^{In(-8)} & e^{In(24)} \end{bmatrix} = \mathcal{M}$ *(say); it is easily seen that ;*

$$(\mathcal{M}^{*2}\mathcal{M}^2)(\mathcal{M}+\mathcal{M}^*) = \begin{bmatrix} e^{In(-491520)} & e^{In(-2719744)} \\ e^{In(1015808)} & e^{In(5603328)} \end{bmatrix}$$ *and*

$$(\mathcal{M}+\mathcal{M}^*)(\mathcal{M}^*\mathcal{M})^2 = \begin{bmatrix} e^{In(-1376256)} & e^{In(5832704)} \\ e^{In(-7929856)} & e^{In(33619968)} \end{bmatrix}$$ *implies* $(\mathcal{M}^{*2}\mathcal{M}^2)(\mathcal{M}+\mathcal{M}^*) \neq (\mathcal{M}+\mathcal{M}^*)(\mathcal{M}^*\mathcal{M})^2$ *hence does not preserve similarity .*

Theorem 4.1.3. *If* $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ *, then* $(\mathcal{T}^2\mathcal{T}^{*2})(\mathcal{T}^* + \mathcal{T}) = (\mathcal{T}^* + \mathcal{T})(\mathcal{T}\mathcal{T}^*)^2$.

Proof. Since $\mathcal{T} \in [V]$, then by Theorem 4.1.2 so is $\mathcal{T}^*$;

thus ;

$((\mathcal{T}^*)^{*2}(\mathcal{T}^*)^2)((\mathcal{T}^*)^* + \mathcal{T}^*) = (\mathcal{T}^* + (\mathcal{T}^*)^*)(\mathcal{T}^*)^*\mathcal{T}^*)^2$

implies that $(\mathcal{T}^2\mathcal{T}^{*2})(\mathcal{T}^* + \mathcal{T}) = (\mathcal{T}^* + \mathcal{T})(\mathcal{T}\mathcal{T}^*)^2$. □

Theorem 4.1.4. *Let* $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ *, it follows ;*

(a) $\mathcal{T} + \mathcal{T}^* \in [V]$.

(b) $(\mathcal{T}^*\mathcal{T})^2 \in [V]$.

(c) $\mathcal{T}^{*2}\mathcal{T}^2 \in [V]$.

(d) $I + \mathcal{T}^{*2}\mathcal{T}^2$, $I + (\mathcal{T}^*\mathcal{T})^2 \in [V]$.

Proof. (a). Let $\mathcal{R} = \mathcal{T} + \mathcal{T}^*$,

implies $\mathcal{R}^* = (\mathcal{T} + \mathcal{T}^*)^*$

= $\mathcal{T}^* + \mathcal{T}^{**} = \mathcal{T}^* + \mathcal{T} = \mathcal{T} + \mathcal{T}^* = \mathcal{R}$.

$\mathcal{R}$ is self-adjoint and from Theorem 4.1.2 , $\mathcal{R} \in [V]$.

(b). $(\mathcal{T}^*\mathcal{T})^2 = (\mathcal{T}^*\mathcal{T})^{*2} = (\mathcal{T}^{**}\mathcal{T}^*)^2 = (\mathcal{T}\mathcal{T}^*)^2$

(c). $\mathcal{T}^{*2}\mathcal{T}^2 = (\mathcal{T}^{*2}\mathcal{T}^2)^* = \mathcal{T}^{**2}\mathcal{T}^{*2} = \mathcal{T}^2\mathcal{T}^{*2} = \mathcal{T}^{*2}\mathcal{T}^2$

(d). $(I + \mathcal{T}^{*2}\mathcal{T}^2) = (I + \mathcal{T}^{*2}\mathcal{T}^2)^* = I^* + \mathcal{T}^{**2}\mathcal{T}^{*2} = I + \mathcal{T}^2\mathcal{T}^{*2}$

and $I + (\mathcal{T}^*\mathcal{T})^2 = I^* + (\mathcal{T}^*\mathcal{T})^{*2} = I + (\mathcal{T}\mathcal{T}^*)^2$

and the proof for (b) , (c) and (d) follows similarly from Theorem 4.1.2

□

Remark 4.1.2. *In Theorem 3.2.7 , properties of idempotent operators were used to establish a connection between the class of 2-metrically equivalent operators and metrically equivalent operators. Similarly, in this case, we utilize both idempotent and self-adjoint properties to establish a connection between the class [V] and (n,m)-normal operators.*

Theorem 4.1.5. *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be both self-adjoint and idempotent , if $\mathcal{T} \in [V]$, then it's an (n,m)-normal operator.*

Proof. suppose $\mathcal{T} \in [V]$;

$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$

$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = \mathcal{T}^{*2}\mathcal{T}^2\mathcal{T} + \mathcal{T}^{*2}\mathcal{T}^2\mathcal{T}^*$

= $\mathcal{T}^{*2}\mathcal{T}^2 + \mathcal{T}^{*2}\mathcal{T}\mathcal{T}^*$ (Since $\mathcal{T}$ is idempotent)

= $\mathcal{T}^{*2}\mathcal{T}^2 + \mathcal{T}^{*2}\mathcal{T}^2$ (Since $\mathcal{T}$ is self-adjoint)

$$= 2\mathcal{T}^{*2}\mathcal{T}^2 \tag{4.14}$$

Similarly ; $(\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$;

$(\mathcal{T}^*\mathcal{T})^2\mathcal{T} + (\mathcal{T}^*\mathcal{T})^2\mathcal{T}^*$

= $\mathcal{T}^{*2}\mathcal{T}^2\mathcal{T} + \mathcal{T}^{*2}\mathcal{T}^2\mathcal{T}^*$

= $\mathcal{T}^{*2}\mathcal{T}^2 + \mathcal{T}^{*2}\mathcal{T}^2$ (Since $\mathcal{T}$ is idempotent and self-adjoint)

= $\mathcal{T}^2\mathcal{T}^{*2} + \mathcal{T}^2\mathcal{T}^{*2}$

$$= 2\mathcal{T}^2\mathcal{T}^{*2} \tag{4.15}$$

From 4.14 and 4.15 we have ;

$\mathcal{T}^{*2}\mathcal{T}^2$ = $\mathcal{T}^2\mathcal{T}^{*2}$ thus $\mathcal{T}$ is an (n,m)-normal operator ; specifically a (2,2)-normal operator.

□

Theorem 4.1.6. *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be a Quasi-Isometry and an Isometry, then $\mathcal{T} \in [V]$.*

Proof. By definition $\mathcal{T}^{*2}\mathcal{T}^2 = \mathcal{T}^*\mathcal{T} = I$;

then

$$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = \mathcal{T}^*\mathcal{T}(\mathcal{T} + \mathcal{T}^*) = I(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*) \tag{4.16}$$

and ;

$$(\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2 = (\mathcal{T} + \mathcal{T}^*)(I)^2 = (\mathcal{T} + \mathcal{T}^*)I = (\mathcal{T} + \mathcal{T}^*) \tag{4.17}$$

4.16 and 4.17 points to $\mathcal{T} \in [V]$. □

Theorem 4.1.7. *Let $\mathcal{T} \in [V]$, if $\mathcal{T}$ is a class (Q) and Quasi-Isometry , then it is a θ-operator.*

Proof. By definition ;

$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$

Since $\mathcal{T}$ is a Quasi-Isometry , then ;

$\mathcal{T}^*\mathcal{T}(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)\mathcal{T}^*\mathcal{T}$ as required. □

Remark 4.1.3. *The result below establishes a correlation between quasi-p-normal operators and skew quasi-p-class (Q) operators. The result unveils the connection between these two classes of operators, shedding light on how quasi-p-normal operators are associated with the skew quasi-p-class (Q).*

Theorem 4.1.8. *If $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ is quasi-p-normal operator , then it is in $[V]$.*

Proof. Let $\mathcal{T}$ be quasi-p-normal , then ;

$$(\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T}) = (\mathcal{T}^*\mathcal{T})(\mathcal{T} + \mathcal{T}^*) \tag{4.18}$$

$$\mathcal{T}\mathcal{T}^*\mathcal{T} + \mathcal{T}^{*2}\mathcal{T} = \mathcal{T}^*\mathcal{T}^2 + \mathcal{T}^{*2}\mathcal{T} \tag{4.19}$$

$$\mathcal{T}^*\mathcal{T}^2 + \mathcal{T}^{*2}\mathcal{T} = \mathcal{T}^*\mathcal{T}^2 + \mathcal{T}^{*2}\mathcal{T} \tag{4.20}$$

pre-multiplying $\mathcal{T}\mathcal{T}^*$ and post-multiplying $\mathcal{T}\mathcal{T}^*$ on both sides ;

$$\mathcal{T}^{*2}\mathcal{T}^3 + \mathcal{T}^{*3}\mathcal{T}^2 = \mathcal{T}^{*2}\mathcal{T}^3 + \mathcal{T}^{*3}\mathcal{T}^2 \tag{4.21}$$

$$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^{*2}\mathcal{T}^2) \tag{4.22}$$

$$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2 \tag{4.23}$$

hence $\mathcal{T} \in [V]$. □

Definition 4.1.2. *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ such that it satisfies $(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T}^n + \mathcal{T}^{*n}) = (\mathcal{T}^n + \mathcal{T}^{*n})(\mathcal{T}^*\mathcal{T})^2$ where n is a positive integer , then $\mathcal{T}$ is a skew quasi-n-p-class (Q) operator .*

Theorem 4.1.9. *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$, if ξ commutes with $\mathcal{C}$ and ζ commutes with with $\mathcal{D}$ and $\xi^2\mathcal{T} = \mathcal{T}\xi^2$, then $\mathcal{T}$ is skew quasi-p-class (Q) where $\xi^2 = (\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*)$, $\zeta^2 = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^{*2}\mathcal{T}^2)$, $\mathcal{C}$ = Re($\mathcal{T}$) = $\frac{\mathcal{T}+\mathcal{T}^*}{2}$ and $\mathcal{D}$ = Im($\mathcal{T}$) = $\frac{\mathcal{T}-\mathcal{T}^*}{2i}$.*

Proof. Since $\xi\mathcal{C} = \mathcal{C}\,\xi$, $\zeta\,\mathcal{D} = \mathcal{D}\,\zeta$; Then $\xi^2\,\mathcal{C} = \mathcal{C}\xi^2$ and $\zeta^2\mathcal{D} = \mathcal{D}\zeta^2$, so ;

$$\xi^2\mathcal{T} + \xi^2\mathcal{T}^* = \mathcal{T}\xi^2 + \mathcal{T}^*\xi^2 \tag{4.24}$$

$$\xi^2\mathcal{T} - \xi^2\mathcal{T}^* = \mathcal{T}\xi^2 - \xi^2\mathcal{T}^* \tag{4.25}$$

$$\mathcal{T}\xi^2 = \xi^2\mathcal{T} \tag{4.26}$$

$$\mathcal{T}(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = \mathcal{T}(\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2 \tag{4.27}$$

$$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T}^2 + \mathcal{T}^{*2}) = (\mathcal{T}^2 + \mathcal{T}^{*2})(\mathcal{T}^*\mathcal{T})^2 \tag{4.28}$$

Similarly with $\zeta^2\mathcal{D}$ =$\mathcal{D}\,\zeta^2$, we get ;

$$\zeta^2\mathcal{T} - \zeta^2\mathcal{T}^* = \mathcal{T}\zeta^2 - \mathcal{T}^*\zeta^2 \tag{4.29}$$

$$\mathcal{T}\zeta^2 = \zeta^2\mathcal{T} \tag{4.30}$$

$$\mathcal{T}(\mathcal{T}^*\mathcal{T})^2(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2\mathcal{T} \tag{4.31}$$

$$(\mathcal{T}^*\mathcal{T})^2(\mathcal{T}^2 + \mathcal{T}^{*2}) = (\mathcal{T}^2 + \mathcal{T}^{*2})(\mathcal{T}^{*2}\mathcal{T}^2) \tag{4.32}$$

$$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T}^2 + \mathcal{T}^{*2}) = (\mathcal{T}^2 + \mathcal{T}^{*2})(\mathcal{T}^*\mathcal{T})^2 \tag{4.33}$$

hence $\mathcal{T}$ is skew quasi-2-p-class (Q) operator. □

Theorem 4.1.10. *Let $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathcal{B}(\mathcal{H})$ be skew quasi-p-class (Q) operators such that $\mathcal{T}_1^{*2}\ \mathcal{T}_2^2 = \mathcal{T}_2^{*2}\ \mathcal{T}_1^2 = \mathcal{T}_1^{*2}\ \mathcal{T}_1 = \mathcal{T}_2^{*2}\ \mathcal{T}_1 = \mathcal{T}_2^{*2}\ \mathcal{T}_1^* = 0$, then $\mathcal{T}_1 + \mathcal{T}_2$ is skew quasi-p-class (Q) operator.*

Proof. By assumption , $\mathcal{T}_1$ and $\mathcal{T}_2$ are skew quasi-p-class (Q) operators , then ;

$$((\mathcal{T}_1+\mathcal{T}_2)^{*2}(\mathcal{T}_1+\mathcal{T}_2)^2)((\mathcal{T}_1+\mathcal{T}_2)+(\mathcal{T}_1+\mathcal{T}_2)^*) \tag{4.34}$$

$$=((\mathcal{T}_1^{*2}+\mathcal{T}_2^{*2})(\mathcal{T}_1^2+\mathcal{T}_2^2))((\mathcal{T}_1+\mathcal{T}_2)+(\mathcal{T}_1^*+\mathcal{T}_2^*)) \tag{4.35}$$

$$=(\mathcal{T}_1^{*2}+\mathcal{T}_2^{*2})(\mathcal{T}_1^2+\mathcal{T}_2^2))(\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_1^*+\mathcal{T}_2^*) \tag{4.36}$$

$$=(\mathcal{T}_1^{*2}\mathcal{T}_1^2+(\mathcal{T}_1^{*2}\mathcal{T}_2^2+\mathcal{T}_2^{*2}\mathcal{T}_1^2+\mathcal{T}_2^{*2}\mathcal{T}_2)(\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_1^*+\mathcal{T}_2^*) \tag{4.37}$$

Since $\mathcal{T}_1^{*2}$ $\mathcal{T}_2^2$ = $\mathcal{T}_2^{*2}$ $\mathcal{T}_1^2$ = 0

$$(\mathcal{T}_1^{*2}\mathcal{T}_1^2+\mathcal{T}_2^{*2}\mathcal{T}_2)(\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_1^*+\mathcal{T}_2^*) \tag{4.38}$$

$$=\mathcal{T}_1^{*2}\mathcal{T}_1^2\mathcal{T}_1+\mathcal{T}_1^{*2}\mathcal{T}_1^2\mathcal{T}_2+\mathcal{T}_1^{*2}\mathcal{T}_1^2\mathcal{T}_1^*+\mathcal{T}_1^{*2}\mathcal{T}_1^2\mathcal{T}_2^*+\mathcal{T}_2^{*2}\mathcal{T}_2^2+\mathcal{T}_2^{*2}\mathcal{T}_2^2\mathcal{T}_1+\mathcal{T}_2^{*2}\mathcal{T}_2^2\mathcal{T}_2\mathcal{T}_1^*+\mathcal{T}_2^{*2}\mathcal{T}_2^2\mathcal{T}_2^* \tag{4.39}$$

Since $\mathcal{T}_1^{*2}$ $\mathcal{T}_1$ = $\mathcal{T}_2^{*2}$ $\mathcal{T}_1$ = $\mathcal{T}_2^{*2}$ $\mathcal{T}_1^*$=0 ;

$$=\mathcal{T}_1^{*2}\mathcal{T}_2^2\mathcal{T}_1+\mathcal{T}_1^{*2}\mathcal{T}_2^2\mathcal{T}_2+\mathcal{T}_1^{*2}\mathcal{T}_2^2\mathcal{T}_1^*+\mathcal{T}_1^{*2}\mathcal{T}_2^2\mathcal{T}_2^* \tag{4.40}$$

$$=(\mathcal{T}_2^2\mathcal{T}_1+\mathcal{T}_2^2\mathcal{T}_2+\mathcal{T}_2^2\mathcal{T}_1^*+\mathcal{T}_2^2\mathcal{T}_2^*)\mathcal{T}_1^{*2} \tag{4.41}$$

$$=(\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_1^*+\mathcal{T}_2^*)(\mathcal{T}_1^{*2}\mathcal{T}_2^2) \tag{4.42}$$

$$=((\mathcal{T}_1+\mathcal{T}_2)+(\mathcal{T}_1+\mathcal{T}_2)^*)(\mathcal{T}_1^*\mathcal{T}_2)^2 \tag{4.43}$$

hence $\mathcal{T}_1$ + $\mathcal{T}_2$ is skew quasi-p-class (Q) operator. □

Corollary 4.1.11. *Let $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathcal{B}(\mathcal{H})$ be skew quasi-p-class (Q) operators such that $\mathcal{T}_1^{*2}$ $\mathcal{T}_2^2$ = $\mathcal{T}_2^{*2}$ $\mathcal{T}_1^2$ = $\mathcal{T}_1^{*2}$ $\mathcal{T}_1$ = $\mathcal{T}_2^{*2}$ $\mathcal{T}_1$ = $\mathcal{T}_2^{*2}$ $\mathcal{T}_1^*$ = 0 , then $\mathcal{T}_1$ - $\mathcal{T}_2$ is skew quasi-p-class (Q) operator.*

Proof. The proof follows directly from Theorem 4.1.10. □

Theorem 4.1.12. *Let $\mathcal{T}=\mathcal{U}|\mathcal{T}|$ be the polar decomposition of $\mathcal{T}\in\mathcal{B}(\mathcal{H})$, then $\mathcal{T}\in[V]$ if $|\mathcal{T}|\mathcal{U}=\mathcal{U}|\mathcal{T}|$.*

Proof. Let $\mathcal{T} \in [V]$, then ,

$$[(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*)] - [(\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2] = 0 \tag{4.44}$$

$$= ((\mathcal{U}^*|\mathcal{T}|^2\mathcal{U}|\mathcal{T}|^2)(\mathcal{U}|\mathcal{T}| + \mathcal{U}^*|\mathcal{T}|)) - ((\mathcal{U}|\mathcal{T}| + \mathcal{U}^*|\mathcal{T}|)(\mathcal{U}^*|\mathcal{T}|\mathcal{U}|\mathcal{T}|)^2) \tag{4.45}$$

$$= (|\mathcal{T}|^2(\mathcal{U}^*\mathcal{U}|\mathcal{T}|^2)(\mathcal{U}|\mathcal{T}| + \mathcal{U}^*|\mathcal{T}|)) - ((\mathcal{U}|\mathcal{T}| + \mathcal{U}^*|\mathcal{T}|)(|\mathcal{T}|\mathcal{U}^*\mathcal{U}|\mathcal{T}|)^2) \tag{4.46}$$

$$= (|\mathcal{T}|^4((\mathcal{U}|\mathcal{T}| + \mathcal{U}^*|\mathcal{T}|)) - ((\mathcal{U}|\mathcal{T}| + \mathcal{U}^*|\mathcal{T}|)|\mathcal{T}|^4) \tag{4.47}$$

$$= \mathcal{U}|\mathcal{T}|^5 + \mathcal{U}^*|\mathcal{T}|^5 - \mathcal{U}|\mathcal{T}|^5 - \mathcal{U}^*|\mathcal{T}|^5 = 0. \tag{4.48}$$

hence we get ; $(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$ and thus $\mathcal{T} \in [V]$. □

Theorem 4.1.13. *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be $\mathcal{T} \in [V]$, then $\mathcal{T}$ is a class (Q) if its unitary.*

Proof. $\mathcal{T}$ being skew quasi-p-class (Q) , implies ;

$\mathcal{T}^{*2}\mathcal{T}^2(\mathcal{T} + \mathcal{T}^*)$

$$= (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2. \tag{4.49}$$

pre-multiplying and post-multiplying 4.49 by $\mathcal{T}^*$ and $\mathcal{T}$ respectively we have ;

$\mathcal{T}^{*2}\mathcal{T}^2(\mathcal{T}^*\mathcal{T} + \mathcal{T}\mathcal{T}^*) = (\mathcal{T}^*\mathcal{T} + \mathcal{T}^*\mathcal{T})(\mathcal{T}^*\mathcal{T})^2$

$\mathcal{T}^{*2}\mathcal{T}^2 = (\mathcal{T}^*\mathcal{T})^2$ as desired. □

Theorem 4.1.14. *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be $\mathcal{T} \in [V]$. If $\mathcal{T}$ is both 2-self-adjoint and self-adjoint it follows it's an n-quasinormal operator.*

Proof. By definition ;

$\mathcal{T}^{*2}\mathcal{T}^2(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2.$

Now suppose $\mathcal{T}$ is both 2-self adjoint and self adjoint , then ;

$\mathcal{T}^{*2} = \mathcal{T}^2 = \mathcal{T}^* = \mathcal{T}$; hence ;

$(\mathcal{T}^*\mathcal{T})\mathcal{T}^2 = \mathcal{T}^2(\mathcal{T}^{*2}\mathcal{T}^2)$

$(\mathcal{T}^*\mathcal{T})\mathcal{T}^2 = \mathcal{T}^2(\mathcal{T}^*\mathcal{T})$

$\mathcal{T}^2(\mathcal{T}^*\mathcal{T}) = (\mathcal{T}^*\mathcal{T})\mathcal{T}^2$. Hence $\mathcal{T}$ is an n-quasinormal operator for n=2. □

Theorem 4.1.15. *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be a unitary operator such that $\mathcal{T} \in [V]$. If $\mathcal{T}$ is both a quasi-isometry and self-adjoint , then it's an n-normal operator.*

Proof. Since $\mathcal{T} \in [V]$ we have ;

$(\mathcal{T}^{*2}\mathcal{T}^2)(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$

by Theorem 4.1.13 we have ;

$\mathcal{T}^{*2}\mathcal{T}^2(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)\mathcal{T}^{*2}\mathcal{T}^2$.

suppose $\mathcal{T}$ is a quasi-isometry , then ;

$\mathcal{T}^*\mathcal{T}(\mathcal{T} + \mathcal{T}^*) = (\mathcal{T} + \mathcal{T}^*)\mathcal{T}^*\mathcal{T}$

similarly , if $\mathcal{T}$ is self-adjoint then we have ;

$\mathcal{T}^*\mathcal{T}(\mathcal{T} + \mathcal{T}) = (\mathcal{T} + \mathcal{T})\mathcal{T}^*\mathcal{T}$

$2\mathcal{T}^*\mathcal{T}^2 = 2\mathcal{T}^2\mathcal{T}^*$

$\mathcal{T}^*\mathcal{T}^2 = \mathcal{T}^2\mathcal{T}^*$ hence $\mathcal{T}$ is an n-normal operator for n=2. □

Corollary 4.1.16. *If $\mathcal{T} \in [V]$ is such that its both a Quasi-Isometry and self-adjoint , then it's isoloid and polaroid.*

Proof. The proof follows from Theorem 3.2.1 and Theorem 4.1.15 respectively . □

Theorem 4.1.17. *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be a self-adjoint skew quasi-p-class (Q) operator . Then $\mathcal{T}$ is* *an (n,p)-quasinormal operator provided it is unitary.*

Proof. By definition ;

$\mathcal{T}^{*2}\mathcal{T}^2(\mathcal{T}+\mathcal{T}^*) = (\mathcal{T}+\mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$

since $\mathcal{T}$ is class (Q) by Theorem 4.1.13 ,

$(\mathcal{T}^*\mathcal{T})^2(\mathcal{T}+\mathcal{T}^*) = (\mathcal{T}+\mathcal{T}^*)(\mathcal{T}^*\mathcal{T})^2$

$\mathcal{T}$ being self-adjoint ensures ,

$\mathcal{T}(\mathcal{T}^*\mathcal{T})^2 = (\mathcal{T}^*\mathcal{T})^2\mathcal{T}$

hence $\mathcal{T}$ is (n,p)-quasinormal operator for n=1 and p=2 .

□

Theorem 4.1.18. *If $\mathcal{T} \in [V]$ such that it's both Quasi-Isometry and self-adjoint , then it has Bishop's property (property β).*

Proof. By Theorem 4.1.15 , $\mathcal{T}$ is an n-normal operator and by Theorem 3.2.2 , the proof follows for $\mathcal{T}$. □

Chapter 5

Posimetrically Equivalent Operators

5.1 Introduction

This chapter focuses on our second objective, which involves studying the class of Posimetrically equivalent operators. We extensively analyze the properties of this class and explore its connections and relationships with other classes of operators. Throughout the chapter, we conduct a thorough examination of Posimetrically equivalent operators, providing a detailed understanding of their distinct characteristics and how they relate to other classes .

Definition 5.1.1. *Two operators $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ are said to be Posimetrically equivalent denoted by $\mathcal{S} \sim_{\mathrm{p}} \mathcal{T}$ if $(\mathcal{S}^*\mathcal{S})\ (\mathcal{S} + \mathcal{S}^*) = (\mathcal{T}^*\mathcal{T})\ (\mathcal{T} + \mathcal{T}^*)$.*

Remark 5.1.1. *Similar to Theorem 3.2.8 and Corollary 3.2.9, where (n,m)-normality was established for the class of (n,m)-metrically equivalent operators, we also establish quasi-p-normality for the class of posimetrically equivalent operators in the subsequent results. These results demonstrate the connection between quasi-p-normality and posimetric equivalence, providing insights into the properties and behaviors of operators within these operator classes.*

Theorem 5.1.1. *If $\mathcal{T}$ is quasi-p-normal and $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ is unitarily equivalent to $\mathcal{T}$, then $\mathcal{S}$ is quasi-p-normal.*

Proof. Suppose $\mathcal{S} = \mathcal{U}^*\mathcal{T}\mathcal{U}$ where $\mathcal{U}$ is unitary and $\mathcal{S}$ is quasi-p-normal , then

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = ((\mathcal{U}^*\mathcal{T}^*\mathcal{U})(\mathcal{U}^*\mathcal{T}\mathcal{U}))(\mathcal{U}^*\mathcal{T}\mathcal{U}+\mathcal{U}^*\mathcal{T}^*\mathcal{U}) \tag{5.1}$$

$$= ((\mathcal{U}^*\mathcal{T}^*\mathcal{U}\mathcal{U}^*\mathcal{T}\mathcal{U})(\mathcal{U}^*\mathcal{T}\mathcal{U}+\mathcal{U}^*\mathcal{T}^*\mathcal{U}) \tag{5.2}$$

$$= (\mathcal{U}^*\mathcal{T}^*\mathcal{T}\mathcal{U})(\mathcal{U}^*\mathcal{T}\mathcal{U}+\mathcal{U}^*\mathcal{T}^*\mathcal{U}) \tag{5.3}$$

$$= (\mathcal{U}^*\mathcal{T}\mathcal{U}+\mathcal{U}^*\mathcal{T}^*\mathcal{U})(\mathcal{U}^*\mathcal{T}^*\mathcal{T}\mathcal{U}) \tag{5.4}$$

$$= (\mathcal{S}\mathcal{U}^*\mathcal{U}+\mathcal{S}^*\mathcal{U}^*\mathcal{U})(\mathcal{S}^*\mathcal{U}^*\mathcal{U}\mathcal{S}) \tag{5.5}$$

$$= (\mathcal{S}+\mathcal{S}^*)(\mathcal{S}^*\mathcal{S}). \tag{5.6}$$

hence the proof. □

Corollary 5.1.2. *An operator $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ is quasi-p-normal if and only if $\mathcal{S}$ and $\mathcal{S}^*$ are Posimetrically equivalent.*

Proof. The proof follows from Theorem 5.1.1 □

Proposition 5.1.1. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(\mathcal{H})$ be bounded linear operators with $\mathcal{S} \sim_p \mathcal{T}$, then ;*

1. *$\mathcal{S}$ is isometric whenever $\mathcal{T}$ is isometric*
2. *$\mathcal{S}$ is a contraction whenever $\mathcal{T}$ is a contraction*
3. *$\lambda\mathcal{S}$ and $\lambda\mathcal{T}$ are Posimetrically equivalent for any $\lambda \in \mathbb{R}$*
4. *The restriction $\mathcal{S}/\mathcal{M}$ of $\mathcal{S}$ and $\mathcal{T}/\mathcal{M}$ of $\mathcal{T}$ to any closed subspace $\mathcal{M}$ of $\mathcal{H}$ that reduces $\mathcal{S}$*

Proof. Proof for 1 and 2 is trivial ;

for 3 ; since $\mathcal{S}$ and $\mathcal{T}$ are Posimetrically equivalent we have ;

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = (\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*) \tag{5.7}$$

$$= ((\lambda\mathcal{S})^*(\lambda\mathcal{S}))(\lambda\mathcal{S}+(\lambda\mathcal{S})^*) = ((\lambda\mathcal{T})^*(\lambda\mathcal{T}))(\lambda\mathcal{T}+(\lambda\mathcal{T})^*) \tag{5.8}$$

$$= \lambda^2(\mathcal{S}^*\mathcal{S})\lambda(\mathcal{S}+\mathcal{S}^*) = \lambda^2(\mathcal{T}^*\mathcal{T})\lambda(\mathcal{T}+\mathcal{T}^*)\ since\ \lambda \in \mathbb{R}, \lambda = \lambda^* \tag{5.9}$$

$$= \lambda^3(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = \lambda^3(\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*) \tag{5.10}$$

from 5.7 and 5.10 $\lambda\mathcal{S}$ and $\lambda\mathcal{T}$ are Posimetrically equivalent.

For 4 ;

$$((\mathcal{S}/\mathcal{M})^*(\mathcal{S}/\mathcal{M}))((\mathcal{S}/\mathcal{M})+(\mathcal{S}/\mathcal{M})^*) = ((\mathcal{T}/\mathcal{M})^*(\mathcal{T}/\mathcal{M}))((\mathcal{T}/\mathcal{M})+(\mathcal{T}/\mathcal{M})^*) \tag{5.11}$$

$$(\mathcal{S}^*\mathcal{S}/\mathcal{M})(\mathcal{S}/\mathcal{M}+\mathcal{S}^*/\mathcal{M}) = (\mathcal{T}^*\mathcal{T}/\mathcal{M})(\mathcal{T}/\mathcal{M}+\mathcal{T}^*/\mathcal{M}) \tag{5.12}$$

$$((\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*))/\mathcal{M} = ((\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*))/\mathcal{M} \tag{5.13}$$

$$(\mathcal{S}^*\mathcal{S})/\mathcal{M}(\mathcal{S}+\mathcal{S}^*)/\mathcal{M} = (\mathcal{T}^*\mathcal{T})/\mathcal{M}(\mathcal{T}+\mathcal{T}^*)/\mathcal{M} \tag{5.14}$$

$$((\mathcal{S}^*)/\mathcal{M}(\mathcal{S})/\mathcal{M})((\mathcal{S}/\mathcal{M})+(\mathcal{S}^*/\mathcal{M})) = ((\mathcal{T}^*/\mathcal{M})(\mathcal{T})/\mathcal{M}))((\mathcal{T}/\mathcal{M})+(\mathcal{T}^*/\mathcal{M})) \tag{5.15}$$

□

Theorem 5.1.3. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(\mathcal{H})$ be posimetrically equivalent. If $\mathcal{S}$ and $\mathcal{T}$ are complex symmetric operators , then $(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = (\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*)$ holds.*

Proof. If $\mathcal{S}$ and $\mathcal{T}$ are complex symmetric operators , then ; $\mathcal{S}^* = \mathcal{C}\mathcal{S}\mathcal{C}$, $\mathcal{S} = \mathcal{C}\mathcal{S}^*\mathcal{C}$ and $\mathcal{T}^* =$

$\mathcal{CTC}$, $\mathcal{T} = \mathcal{CT}^*\mathcal{C}$ with $\mathcal{C}^2 = I$. It then implies that

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = (\mathcal{CSCCS}^*\mathcal{C})(\mathcal{CS}^*\mathcal{C}+\mathcal{CSC}) \tag{5.16}$$

$$= (\mathcal{CSS}^*\mathcal{C})(\mathcal{CS}^*+\mathcal{SC}) \tag{5.17}$$

$$= (\mathcal{CS}^*\mathcal{SC})(\mathcal{CS}+\mathcal{S}^*\mathcal{C}) \tag{5.18}$$

$$= (\mathcal{C}(\mathcal{S}^*\mathcal{S}))((\mathcal{S}+\mathcal{S}^*)\mathcal{C}) \tag{5.19}$$

$$= \mathcal{C}^2(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) \tag{5.20}$$

Similarly ;

$$(\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*) = (\mathcal{CTCCT}^*\mathcal{C})(\mathcal{CT}^*\mathcal{C}+\mathcal{CTC}) \tag{5.21}$$

$$= (\mathcal{CTT}^*\mathcal{C})(\mathcal{CT}^*+\mathcal{TC}) \tag{5.22}$$

$$= (\mathcal{CT}^*\mathcal{TC})(\mathcal{CT}+\mathcal{T}^*\mathcal{C}) \tag{5.23}$$

$$= (\mathcal{C}(\mathcal{T}^*\mathcal{T}))((\mathcal{T}+\mathcal{T}^*)\mathcal{C}) \tag{5.24}$$

$$= \mathcal{C}^2(\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*) \tag{5.25}$$

From 5.20 and 5.25 we get ;

$$\mathcal{C}^2(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = \mathcal{C}^2(\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*) \tag{5.26}$$

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = (\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*) \tag{5.27}$$

As required. □

Proposition 5.1.2. *Let $\mathcal{S},\mathcal{T} \in \mathcal{B}(\mathcal{H})$, such that $\zeta = (\mathcal{S}^*\mathcal{S})$ - $(\mathcal{S}+\mathcal{S}^*)$,$\vartheta = (\mathcal{S}^*\mathcal{S}) + (\mathcal{S}+\mathcal{S}^*)$ and $\eta = (\mathcal{T}^*\mathcal{T})$ - $(\mathcal{T}+\mathcal{T}^*)$,$\psi = (\mathcal{T}^*\mathcal{T}) + (\mathcal{T}+\mathcal{T}^*)$. Then $\mathcal{S}$ and $\mathcal{T}$ are Posimetrically equivalent if $\zeta\vartheta = \eta\psi$.*

Proof.

$$\zeta\vartheta = \eta\psi. \tag{5.28}$$

$$\zeta\vartheta = ((\mathcal{S}^*\mathcal{S}) - (\mathcal{S} + \mathcal{S}^*))((\mathcal{S}^*\mathcal{S}) + (\mathcal{S} + \mathcal{S}^*)). \tag{5.29}$$

A simple computation of 5.29 gives us ; $(\mathcal{S}^*\mathcal{S})(\mathcal{S} + \mathcal{S}^*)$

$$\eta\psi = ((\mathcal{T}^*\mathcal{T}) - (\mathcal{T} + \mathcal{T}^*))((\mathcal{T}^*\mathcal{T}) + (\mathcal{T} + \mathcal{T}^*)). \tag{5.30}$$

A simple computation of 5.30 gives us ; $(\mathcal{T}^*\mathcal{T})(\mathcal{T} + \mathcal{T}^*)$

From 5.29 and 5.30

$$\zeta\vartheta = \eta\psi \tag{5.31}$$

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S} + \mathcal{S}^*) = (\mathcal{T}^*\mathcal{T})(\mathcal{T} + \mathcal{T}^*) \tag{5.32}$$

hence $\mathcal{S}$ and $\mathcal{T}$ are Posimetrically equivalent

□

Theorem 5.1.4. *If $\mathcal{S}$ and $\mathcal{T}$ are Posimetrically equivalent with polar decompositions $\mathcal{S} = \mathcal{U}|\ \mathcal{S}\ |$ and $\mathcal{T} = \mathcal{U}|\ \mathcal{T}|$, then $|\ \mathcal{S}\ |^3 = |\ \mathcal{T}\ |^3$ if and only if $\mathcal{U}|\mathcal{S}| = |\mathcal{S}|\mathcal{U}$ and $\mathcal{U}|\mathcal{T}| = |\mathcal{T}|\mathcal{U}$.*

Proof. Since $\mathcal{S}$ and $\mathcal{T}$ are Posimetrically equivalent ;

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = (\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*) \tag{5.33}$$

$$(\mathcal{U}^* \mid \mathcal{S} \mid \mathcal{U} \mid \mathcal{S} \mid)(\mathcal{U} \mid \mathcal{S} \mid +\mathcal{U}^* \mid \mathcal{S} \mid) = (\mathcal{U}^* \mid \mathcal{T} \mid \mathcal{U} \mid \mathcal{T} \mid)(\mathcal{U} \mid \mathcal{T} \mid +\mathcal{U}^* \mid \mathcal{T} \mid) \tag{5.34}$$

$$(\mid \mathcal{S} \mid \mathcal{U}^*\mathcal{U} \mid \mathcal{S} \mid)(\mathcal{U} \mid \mathcal{S} \mid +\mathcal{U}^* \mid \mathcal{S} \mid) = (\mid \mathcal{T} \mid \mathcal{U}^*\mathcal{U} \mid \mathcal{T} \mid)(\mathcal{U} \mid \mathcal{T} \mid +\mathcal{U}^* \mid \mathcal{T} \mid) \tag{5.35}$$

$$(\mid \mathcal{S} \mid^2)(\mathcal{U} \mid \mathcal{S} \mid +\mathcal{U}^* \mid \mathcal{S} \mid) = (\mid \mathcal{T} \mid^2)(\mathcal{U} \mid \mathcal{T} \mid +\mathcal{U}^* \mid \mathcal{T} \mid) \tag{5.36}$$

$$U \mid \mathcal{S} \mid^3 +\mathcal{U}^* \mid \mathcal{S} \mid^3 = \mathcal{U} \mid \mathcal{T} \mid^3 +\mathcal{U}^* \mid \mathcal{T} \mid^3 \tag{5.37}$$

$$\mathcal{U} \mid \mathcal{S} \mid^3 + \mid \mathcal{S} \mid^3 \mathcal{U}^* = \mathcal{U} \mid \mathcal{T} \mid^3 + \mid \mathcal{T} \mid^3 \mathcal{U}^* \tag{5.38}$$

pre-multiplying both the left and right hand side of 5.38 by $\mathcal{U}^*$ and post-multiplying the same by $\mathcal{U}$;

$$\mathcal{U}^*\mathcal{U} \mid \mathcal{S} \mid^3 + \mid \mathcal{S} \mid^3 \mathcal{U}^*\mathcal{U} = \mathcal{U}^*\mathcal{U} \mid \mathcal{T} \mid^3 + \mid \mathcal{T} \mid^3 \mathcal{U}^*\mathcal{U} \tag{5.39}$$

$$2 \mid \mathcal{S} \mid^3 = 2 \mid \mathcal{T} \mid^3 \tag{5.40}$$

$$\mid \mathcal{S} \mid^3 = \mid \mathcal{T} \mid^3 \tag{5.41}$$

hence the proof. □

5.1.1 The correlation between Posimetric equivalence and other equivalence relations.

Remark 5.1.2. *The following result establishes the connection between Posimetric equivalence and 2-metric equivalence, which is a subclass of n-metrically equivalent operators. Specifically, the result shows that if two operators are Posimetrically equivalent and self-adjoint, then they are also 2-metrically equivalent. This highlights the relationship between these two types* *of equivalence and emphasizes the condition of self-adjointness in establishing the 2-metric equivalence property.*

Theorem 5.1.5. *If $\mathcal{S}, \mathcal{T} \in \mathcal{B}(\mathcal{H})$ are Posimetrically equivalent ,then they are 2-metrically equivalent provided $\mathcal{S}$ and $\mathcal{T}$ are self-adjoint.*

Proof. By assumption ;

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S} + \mathcal{S}^*) = (\mathcal{T}^*\mathcal{T})(\mathcal{T} + \mathcal{T}^*) \tag{5.42}$$

since $\mathcal{S}$ and $\mathcal{T}$ are self-adjoint ;

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S} + \mathcal{S}) = (\mathcal{T}^*\mathcal{T})(\mathcal{T} + \mathcal{T}) \tag{5.43}$$

$$2(\mathcal{S}^*\mathcal{S})\mathcal{S} = 2(\mathcal{T}^*\mathcal{T})\mathcal{T} \tag{5.44}$$

$$\mathcal{S}^*\mathcal{S}^2 = \mathcal{T}^*\mathcal{T}^2 \tag{5.45}$$

□

Remark 5.1.3. *The following result establishes the connection between the class of Posimetrically equivalent operators and the class of metrically equivalent operators. Specifically, it states that if two operators are Posimetrically equivalent and also idempotent, then they are also metrically equivalent. This result highlights the relationship between these two classes of operators and emphasizes the condition of idempotence in establishing the metric equivalence property.*

Theorem 5.1.6. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(\mathcal{H})$ be Posimetrically equivalent , then they are metrically equivalent provided they are idempotent.*

Proof. Since $\mathcal{S}$ and $\mathcal{T}$ are Posimetrically equivalent , we have ;

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = (\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*) \tag{5.46}$$

$$(\mathcal{S}^*\mathcal{S}\mathcal{S}) + (\mathcal{S}^*\mathcal{S}\mathcal{S}^*) = (\mathcal{T}^*\mathcal{T}\mathcal{T}) + (\mathcal{T}^*\mathcal{T}\mathcal{T}^*) \tag{5.47}$$

$$(\mathcal{S}^*\mathcal{S}^2) + (\mathcal{S}^{*2}\mathcal{S}) = (\mathcal{T}^*\mathcal{T}^2) + (\mathcal{T}^{*2}\mathcal{T}) \tag{5.48}$$

Since $\mathcal{S}$ and $\mathcal{T}$ are idempotent ; $\mathcal{S} = \mathcal{S}\mathcal{S}$, $\mathcal{S}^* = \mathcal{S}^*\mathcal{S}^*$ and $\mathcal{T} = \mathcal{T}\mathcal{T}$, $\mathcal{T}^* = \mathcal{T}^*\mathcal{T}^*$, hence ;

$$(\mathcal{S}^*\mathcal{S}) + (\mathcal{S}^*\mathcal{S}) = (\mathcal{T}^*\mathcal{T}) + (\mathcal{T}^*\mathcal{T}) \tag{5.49}$$

$$2(\mathcal{S}^*\mathcal{S}) = 2(\mathcal{T}^*\mathcal{T}) \tag{5.50}$$

$$\mathcal{S}^*\mathcal{S} = \mathcal{T}^*\mathcal{T}. \tag{5.51}$$

hence the proof. □

Remark 5.1.4. *The result below establishes the relationship between posimetrically equivalent operators and almost similarly equivalent operators.*

Theorem 5.1.7. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(\mathcal{H})$ be two similar Posimetrically equivalent operators , then they are almost similarly equivalent provided they are isometries and $\mathcal{S}+\mathcal{S}^* = \mathcal{S}^*+\mathcal{S}$ and $\mathcal{T}+\mathcal{T}^*$ $= \mathcal{T}^*+\mathcal{T}$.*

Proof. Since $\mathcal{S}$ and $\mathcal{T}$ are similar Posimetrically equivalent , then there exists an invertible operator $\mathcal{N}$ such that ;

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = \mathcal{N}^{-1}((\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*))\mathcal{N} \tag{5.52}$$

$$\mathcal{S}^*\mathcal{S}\mathcal{S}+\mathcal{S}^*\mathcal{S}\mathcal{S}^* = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T}\mathcal{T}+\mathcal{T}^*\mathcal{T}\mathcal{T}^*)\mathcal{N} \tag{5.53}$$

Pre-multiplying and post-multiplying both the left hand side of 5.53 by $\mathcal{S}^*$ and $\mathcal{S}$ and right hand side by $\mathcal{T}^*$ and $\mathcal{T}$ respectively we get ;

$$\mathcal{S}^*\mathcal{S}^*\mathcal{S}\mathcal{S}+\mathcal{S}^*\mathcal{S}\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T}^*\mathcal{T}\mathcal{T}+\mathcal{T}^*\mathcal{T}\mathcal{T}^*\mathcal{T})N \tag{5.54}$$

$$\mathcal{S}^{*2}\mathcal{S}^2+\mathcal{S}^{*2}\mathcal{S}^2 = \mathcal{N}^{-1}(\mathcal{T}^{*2}\mathcal{T}^2+\mathcal{T}^{*2}\mathcal{T}^2)\mathcal{N} \tag{5.55}$$

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}^*\mathcal{S}+\mathcal{S}^*\mathcal{S}) = \mathcal{N}^{-1}((\mathcal{T}^*\mathcal{T})(\mathcal{T}^*\mathcal{T}+\mathcal{T}^*\mathcal{T}))\mathcal{N} \tag{5.56}$$

Since $\mathcal{S}$ and $\mathcal{T}$ are isometries ;

$$\mathcal{S}^*\mathcal{S}(2I) = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})(2I)\mathcal{N} \tag{5.57}$$

$$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N} \tag{5.58}$$

Similarly ;

$$(\mathcal{S}^*\mathcal{S})(\mathcal{S}+\mathcal{S}^*) = \mathcal{N}^{-1}((\mathcal{T}^*\mathcal{T})(\mathcal{T}+\mathcal{T}^*))\mathcal{N} \tag{5.59}$$

Since $\mathcal{S}$ and $\mathcal{T}$ are isometries ;

$$I(\mathcal{S}+\mathcal{S}^*) = \mathcal{N}^{-1}(I(\mathcal{T}+\mathcal{T}^*))\mathcal{N} \tag{5.60}$$

$$\mathcal{S}+\mathcal{S}^* = \mathcal{N}^{-1}(\mathcal{T}+\mathcal{T}^*)\mathcal{N} \tag{5.61}$$

$$\mathcal{S}^*+\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^*+\mathcal{T})\mathcal{N} \tag{5.62}$$

From 5.58 and 5.62 , $\mathcal{S}$ and $\mathcal{T}$ are almost similarly equivalent. □

Chapter 6

Mutually Class (Q) Operators

6.1 Introduction

This chapter centers around our third objective, which involves studying the Mutually Class (Q) operators. We conduct a comprehensive analysis of this operator class, thoroughly exploring its properties and investigating its connections with other classes. Throughout the chapter, we delve deep into the unique characteristics of the Mutually Class (Q) operators and examine how they interact with other operator classes.

Definition 6.1.1. *$\mathcal{G}$ and $\mathcal{P}$ are said to be mutually class (Q) if $(\mathcal{G}^*\mathcal{G})^2 = \mathcal{P}^{*2}\mathcal{P}^2$ and $\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{P}^*\mathcal{P})^2$. We denote this class by $\mathcal{G}m_Q\mathcal{P}$.*

Theorem 6.1.1. *If $\mathcal{G}, \mathcal{P} \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{G}m_Q\mathcal{P}$, then $\mathcal{G}\mathcal{P}$ and $\mathcal{P}\mathcal{G}$ are class (Q) operators .*

Proof. Since $\mathcal{G}m_Q\mathcal{P}$,

$$(\mathcal{G}^*\mathcal{G})^2 = \mathcal{P}^{*2}\mathcal{P}^2 \tag{6.1}$$

$$\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{P}^*\mathcal{P})^2 \tag{6.2}$$

6.1 and 6.2 implies ;

$(\mathcal{G}\mathcal{P})^{*2}(\mathcal{G}\mathcal{P})^2 = (\mathcal{P}\mathcal{G})^{*2}(\mathcal{P}\mathcal{G})^2$

= $\mathcal{P}^{*2}\mathcal{G}^{*2}\mathcal{P}^2\mathcal{G}^2$

= $\mathcal{P}^{*2}\mathcal{P}^2\mathcal{G}^{*2}\mathcal{G}^2$

$= \mathcal{P}^{*2}\mathcal{P}^2(\mathcal{P}^*\mathcal{P})^2$

$= \mathcal{P}^{*2}\mathcal{P}^2\mathcal{P}^{*2}\mathcal{P}^2$

$= (\mathcal{G}^*\mathcal{G})^2(\mathcal{G}^*\mathcal{G})^2$

$= \mathcal{G}^2\mathcal{P}^2\mathcal{G}^{*2}\mathcal{P}^{*2}$

$= \mathcal{G}^{*2}\mathcal{P}^{*2}\mathcal{G}^2\mathcal{P}^2$

$= (\mathcal{G}\mathcal{P})^{*2}(\mathcal{G}\mathcal{P})^2$

$= ((\mathcal{G}\mathcal{P})^*(\mathcal{G}\mathcal{P}))^2.$

hence $\mathcal{G}\mathcal{P}$ is class (Q) and on the same note $\mathcal{P}\mathcal{G}$ is class (Q). □

Theorem 6.1.2. *If $\mathcal{G}, \mathcal{P} \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{G}\mathcal{P} = \mathcal{P}\mathcal{G}$ and $\mathcal{G}m_Q\mathcal{P}$, then $\mathcal{G}$ and $\mathcal{P}$ are class (Q) operators .*

Proof. $\mathcal{G}\mathcal{P} = \mathcal{P}\mathcal{G}$ implies $\mathcal{G}^*\mathcal{P}^* = \mathcal{P}^*\mathcal{G}^*$,

hence $\mathcal{G}^{*2}(\mathcal{P}^*\mathcal{P})^2\mathcal{G}^2 = \mathcal{P}^{*2}$

$$(\mathcal{G}^*\mathcal{G})^2\mathcal{P}^2 \tag{6.3}$$

$\mathcal{G}m_Q\mathcal{P}$ implies ;

$\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{P}^*\mathcal{P})^2$ and $(\mathcal{G}^*\mathcal{G})^2 = \mathcal{P}^{*2}\mathcal{P}^2$, replacing in 6.3 , we obtain ;

$\mathcal{G}^{*2}\mathcal{G}^{*2}\mathcal{G}^2\mathcal{G}^2 = \mathcal{P}^{*2}\mathcal{P}^{*2}\mathcal{P}^2\mathcal{P}^2 = \mathcal{G}^{*2}\mathcal{G}^{*2}\mathcal{G}^2\mathcal{G}^2$

$(\mathcal{G}^*\mathcal{G})^4 = \mathcal{G}^{*2}\mathcal{G}^2(\mathcal{G}^*\mathcal{G})^2$

$(\mathcal{G}^*\mathcal{G})^2 = \mathcal{G}^{*2}\mathcal{G}^2$ implying that $\mathcal{G}$ is class (Q). Proof for $\mathcal{P}$ follows suit . □

Theorem 6.1.3. *If $\mathcal{G}, \mathcal{P} \in \mathcal{B}(\mathcal{H})$ are almost class (Q) such that $\mathcal{G}m_Q\mathcal{P}$, then $\mathcal{G}$ and $\mathcal{P}$ are class (Q).*

Proof. $\mathcal{G}$ being almost class (Q) , we have $\mathcal{G}^{*2}\mathcal{G}^2 \geq (\mathcal{G}^*\mathcal{G})^2$. $\mathcal{G}m_Q\mathcal{P}$ implies $\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{P}^*\mathcal{P})^2$ and $(\mathcal{G}^*\mathcal{G})^2 = \mathcal{P}^{*2}\mathcal{P}^2$,hence $(\mathcal{P}^*\mathcal{P})^2 \geq \mathcal{P}^{*2}\mathcal{P}^2$. $\mathcal{P}$ being almost class (Q) implies $\mathcal{P}^{*2}\mathcal{P}^2 \geq (\mathcal{P}^*\mathcal{P})^2$. Hence $\mathcal{P}^{*2}\mathcal{P}^2 = (\mathcal{P}^*\mathcal{P})^2$ implying $\mathcal{P}$ is class (Q). □

Theorem 6.1.4. *Let $\mathcal{G}, \mathcal{P} \in \mathcal{B}(\mathcal{H})$ be $\mathcal{G}m_Q\mathcal{P}$, then they are 2,2-metrically equivalent if and only if $\mathcal{G}$ and $\mathcal{P}$ are class (Q).*

Proof. Let $\mathcal{G}$ and $\mathcal{P}$ be (2,2)-metrically equivalent , $\mathcal{G}^{*2}\mathcal{G}^2 = \mathcal{P}^{*2}\mathcal{P}^2$. Since $\mathcal{G}m_Q\mathcal{P}$, $\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{P}^*\mathcal{P})^2$ implies $\mathcal{P}^{*2}\mathcal{P}^2 = (\mathcal{P}^*\mathcal{P})^2$ hence $\mathcal{P}$ is class (Q). On the converse , let $\mathcal{G}$ and $\mathcal{P}$ be class (Q) ,then $\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{G}^*\mathcal{G})^2$ and $\mathcal{P}^{*2}\mathcal{P}^2 = (\mathcal{P}^*\mathcal{P})^2$. Since $\mathcal{G}m_Q\mathcal{P}$,we have ; $\mathcal{P}^{*2}\mathcal{P}^2 = (\mathcal{G}^*\mathcal{G})^2 = \mathcal{G}^{*2}\mathcal{G}^2$ hence $\mathcal{G}$ and $\mathcal{P}$ are (2,2)-metrically equivalent . □

Theorem 6.1.5. *Let $\mathcal{G}, \mathcal{P} \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{G}\mathcal{P} = \mathcal{P}\mathcal{G}$; if $\mathcal{G}$ and $\mathcal{P}$ are 2- Isometries and $\mathcal{G}m_Q\mathcal{P}$, then $\mathcal{G}$ and $\mathcal{P}$ are metrically equivalent .*

Proof. $\mathcal{G}$ and $\mathcal{P}$ being 2-Isometries implies ;

$\mathcal{G}^{*2}\mathcal{G}^2 - 2\mathcal{G}^*\mathcal{G} + I = 0$ and $\mathcal{P}^{*2}\mathcal{P}^2 - 2\mathcal{P}^*\mathcal{P} + I = 0$. Since $\mathcal{G}m_Q\mathcal{P}$; $\mathcal{G}^{*2}\mathcal{G}^2 = (\mathcal{P}^*\mathcal{P})^2$ and $\mathcal{P}^{*2}\mathcal{P}^2 = (\mathcal{G}^*\mathcal{G})^2$. Hence ;

$$2\mathcal{G}^*\mathcal{G} = (\mathcal{P}^*\mathcal{P})^2 \tag{6.4}$$

$$2\mathcal{P}^*\mathcal{P} = (\mathcal{G}^*\mathcal{G})^2 \tag{6.5}$$

by Theorem 6.1.2 $\mathcal{G}$ and $\mathcal{P}$ are class (Q) , hence 6.4 and 6.5 leads to

$$2\mathcal{G}^*\mathcal{G} = (\mathcal{P}^*\mathcal{P})^2 = \mathcal{P}^{*2}\mathcal{P}^2 \tag{6.6}$$

and

$$2\mathcal{P}^*\mathcal{P} = (\mathcal{G}^*\mathcal{G})^2 = \mathcal{G}^{*2}\mathcal{G}^2 \tag{6.7}$$

Since they are 2-Isometries , 6.6 and 6.7 leads to ;

$$2\mathcal{G}^*\mathcal{G} = (\mathcal{P}^*\mathcal{P})^2 = 2\mathcal{P}^*\mathcal{P} \tag{6.8}$$

and

$$2\mathcal{P}^*\mathcal{P} = (\mathcal{G}^*\mathcal{G})^2 = 2\mathcal{G}^*\mathcal{G} \tag{6.9}$$

From 6.8 and 6.9 we have that $2\mathcal{G}^*\mathcal{G} = 2\mathcal{P}^*\mathcal{P}$ implying $\mathcal{G}^*\mathcal{G} = \mathcal{P}^*\mathcal{P}$ hence the proof. □

Chapter 7

Conclusions and Recommendations

7.1 Introduction

We infer conclusion and recommendations in this chapter basing on the captured specific objectives for the study and also on the out-turns that we obtained in our study .

7.2 Conclusions

In conclusion, the results presented in this study highlight several important properties of skew quasi-p-class (Q) operators. Firstly, it has been established that these operators are closed under unitary and scalar multiplication, indicating their stability under these operations. Secondly, a bounded operator qualifies as a skew quasi-p-class (Q) operator if it possesses both Quasi-Isometry and Isometry properties. This suggests a strong connection between these operator classes.

Furthermore, it has been observed that if a bounded operator is a skew quasi-p-class (Q) operator, it also falls under the class (Q) operator category when it is unitary. This indicates a relationship between the skew quasi-p-class (Q) operators and the class (Q) operators.

The research findings also reveal that a bounded operator, which meets the criteria of being a skew quasi-p-class (Q) operator while also possessing properties of Quasi-Isometry and self-adjointness, exhibits the characteristics of being isoloid and polaroid. This discovery highlights the intrinsic connection between these particular properties and the skew quasi-p-class (Q) operator category.

Additionally, it has been demonstrated that if a bounded operator is a skew quasi-p-class (Q) operator with Quasi-Isometry and self-adjoint properties, it possesses Bishop's property. This result further emphasizes the significance of these properties in the context of the skew quasi-p-class (Q) operators.

Furthermore, the study shows that if two bounded operators are posimetrically equivalent and self-adjoint, they are also 2-metrically equivalent. This result establishes a relationship between posimetric equivalence, self-adjointness, and 2-metric equivalence.

It has also been established that if two operators are posimetrically equivalent, they are metrically equivalent when they are idempotent, indicating a connection between posimetric equivalence, idempotency, and metric equivalence.

The study also reveals that if two operators are mutually class (Q), their products are also mutually class (Q) operators. This result demonstrates the preservation of the class (Q) property under multiplication.

It has also been found that if two operators are almost class (Q) operators and mutually class (Q), they are classified as class (Q) operators. This result highlights the relationship between almost class (Q) operators and class (Q) operators.

Lastly, the study shows that if two operators are mutually class (Q) and 2-Isometries, they are metrically equivalent. This finding emphasizes the connection between mutual class (Q), 2-Isometries, and metric equivalence.

The study also reveals the following significant findings:

[1] . If two operators are mutually class (Q), they are 2,2-metrically equivalent provided they are class (Q). This result indicates that the mutual class (Q) property guarantees a stronger form of metric equivalence, specifically 2,2-metric equivalence.

[2] . The study demonstrates that if two operators are 2-Isometries and mutually class (Q), they are metrically equivalent. This result establishes a connection between the properties of 2-Isometries and mutual class (Q), indicating that they jointly lead to metric equivalence.

Overall, these results contribute to a deeper understanding of the properties of skew quasi-p-class (Q) , posimetrically equivalent and mutually class (Q) operators and related operator classes, shedding light on their fundamental characteristics and connections in operator theory.

7.3 Recommendations

Based on the newly established classes of operators, namely skew quasi-p-class (Q) operators, posimetrically equivalent operators, and mutually class (Q) operators, these operator properties can potentially be harnessed and applied in the telecommunications industry to alleviate traffic congestion , we therefore highly recommend these classes to be incorporated in the telecommunication industry. This is because by leveraging the closure property, these classes of operators can ensure stability and robustness in the communication network. Additionally, the properties of unitary and scalar multiplication can be utilized to optimize resource allocation and enhance the efficiency of data transmission. Implementing these operator classes in the telecommunications industry has the potential to significantly improve network performance, mitigate congestion issues, and ultimately enhance the overall quality of service for end-users.

We also recommend properties of Skew Quasi-p-class (Q) , Posimetrically equivalent and Mutually class (Q) operators to be adopted in construction of codes in the telecommunication industry to help ease traffic flow .

The analysis of class (Q) operators in the typical Hilbert space has not been spent . In this study , we were able to expand class (Q) operators into the classes of skew quasi-p-class (Q) and mutually class (Q) operators and study their correlation to other classes . It would be of sizeable interest therefore for the following to be explored in the future :

[1]. What properties do class (Q) operators enjoy in the semi-Hilbertian space .

[2]. Is mutually class (Q)operators an equivalence relation ?

[3]. Is there a relation between Posimetrically equivalent operators and nearly equivalent operators ?

References

Adel, S. (2012) . A-normal operators in semi Hilbertian spaces , *Australian Journal of Mathematical Analysis and Applications* , 9(1) , 1-12.

Alzuraiqi, S. A. & Patel, A. B. (2010) . On n-Normal Operators , *General mathematics Notes* , 1(2) , 61-73.

Amjad, H. , Abdul, M. & Laith, K. (2019) . α- almost similar operators , *Tikrit Journal of pure science* , 24 (5) , 2415-1726.

Arora, S. C. & Ramesh, K. (1980) . M-hyponormal operators , *Yokohama mathematical Journal* , 28(1) , 41-44.

Bachman, G. & Narici, L. (2000) . *Functional analysis* , Courier Corporation.

Eiman, H. & Mustafa, A. (2016) . On some generalizations of (n,m)-normal powers operators on Hilbert space , *Journal of progressive research in mathematics* , 7(3) , 2395 -0218.

Furuta, T. (2001). *Invitation to linear operators from matrices and bounded linear operators on a Hilbert space* , Taylor and Francis , London .

Hongliang, Z. & Fei, Z. (2014) . A note on n-perinormal operators , *Acta mathematica scientia* , 34B (1) , 194-198.

Ilyas, M. & Reyaz, A. (2012) . Some classes of operators related to p-hyponormal operator , *Advances in pure mathematics* , 2(9) , 419-422.

Jibril, A. A. (2010) . On Operators for which $T^{*2}(T)^2 = (T^*T)^2$, *International mathematical forum* , 5(46) , 2255-2262.

Jibril, A. A. (2010) . On α-Operators , *Journal of Mathematical Sciences : Advances And Applications* , 5 , 277-287.

Jibril, A. A. (1999) . On mutually Normal operators , *Abhath Al-yarmouk* , 8(1) , 53-60.

Jibril, A. A. (1996) . On almost similar operators , *Arabian Journal for science and Enginnering* , 21 (3) , 443-449.

Jibril, A. A. (2008) . On n-power normal operators , *The Arabian Journal for Science and Engineering* , 33 , 247-251.

Kreyszig, E. (1991) . *Introductory functional analysis with applications* , 17 , John Wiley and Sons.

Manikandan, K. M. & Veluchamy, T. (2018) . On (N+K) power class (Q) Operators in the Hilbert space - I , *International journal of mathematics and technology* , 55 (6) , 450-454.

Manikandan, K. M. & Veluchamy, T. (2018) . On (N+K) power class (Q) Operators in the Hilbert space - II , *International journal of mathematics and technology* , 55 (7) , 455-462 .

Mahmoud , K. S. (2016) . Square-Normal Operator , *British journal of mathematics and computer science* , 19 , 2231-0851.

Meenambika, K. , Seshaiah, C. V & Sivamani, N . (2018) . Skew normal operators acting on a Hilbert space , *International journal of statistics and applied mathematics* , 3 (5) , 128-133.

Mecheri, S. & Braha, N. L. (2012) . Spectrum of n-perinormal operators , *operator Matrices* , 6 , 725-734.

Muneo, C. , Ji, E. L. , Kotairo, T. & Atsushi, U. (2018) . Remarks on n-normal Operators , *Faculty of science and mathematics University of Nis , Serbia* , 32(15) , 5441-5451.

Muneo, C. & Biljana, N. (2018) . Spectral properties of n-Normal Operators , *Faculty of Science and Mathematics , University of Nis , Serbia* , 32:14 , 5063-5069.

Muneo, C. , Dijana, M. , Biljana, N. & Taiga, S. (2019) . Spectral properties of square hyponormal operators , *Faculty of science and mathematics* , 33 (15) , 4845-4854.

Musundi, S. , Sitati, N. , Nzimbi, B. & Murwayi, A . (2013) . On Almost Similarity Operator Equivalence Relation , *IJRRAS ,* 15 , 293-299.

Nzimbi, B. M. , Pokhariyal, G. P. & Moindi, S. K. (2013) . A Note on Metric Equivalence of some Operators , *Far East J. F Math. Sci . (FJMS)* , 75(2) , 301-318.

Ould, A .M. S. A. (2014) . The single valued extension property and Bishops property for certain generalized classes of operators on Hilbert spaces , *International Journal of pure and applied mathematics* , 95(3) , 427-452.

Panayappan, S. & Sivamani, N. (2012) . A-Quasi normal Operators in semi- Hilbertian spaces , *Gen . Math . Notes* , 10 (2) , 535-559.

Paramesh, S. , Hemalatha, D. & Nirmala, V. J. (2019) . A study On n-power class (Q) operators , *International research journal of engineering and technology* , 6(1) , 2395-0056.

Revathi, V. & Maheswari, P . (2019) . On Quasi-class (Q) operator , *International journal of mathematics and its applications* , 7 (3) , 47-53.

Revathi, V. & Maheswari, P. (2019) . A study on properties of M quasi- class (Q) operator , *International Journal of advance research , ideas and innovations in technology* , 5 (5) , 387-390.

Sadoon, I. O . (1996) . Nearly equivalent operators , *math . Bohem* , 121 (2) , 133-141.

Senthilkumar, D. & Revathi, N. (2019) . Quasi-p- normal and Quasi-n-p-normal composition, weighted composition and composite multiplication operators , *International journal of scientific and engineering Research* , 10 (1) , 2229-5518.

Senthilkumar , D. , Maheswari, N. & Santhi, R. (2012) . K-Quasi-normal operators , *International Journal of mathematics and computation* , 15(2) , 0974-5718 .

Stella, I. M & Vijayalakshmi, P. (2015) . Fuglede-putnam Theorem and quasi-nilpotent part of n-power normal operators , *Tamkang Journal of mathematics* , 46 (2) , 151-165.

Schoichi, O. (2002) . On some classes of q-deformed Operators , *J.operator theory* , 48 (A65) , 151-186.

Veluchamy, T. & Manikandan, K . M . (2016) . N-Power quasi normal operators on the Hilbert space , *IOSR Journal of Mathematics* , 12(1) , 06-09.

Wanjala, V. , Obogi, R. K. & Okoya, M. O. (2020) . On N-Metric equivalence of Operators , *International Journal of Mathematics and its Applications* , 8(1) , 107-109.

Wanjala, V. & Adhiambo, O . (2021) . On class (BQ) operators , *Global Journal of Advanced research* , 8(4) , 1-3.

Wanjala, V. & Adhiambo, O . (2021) . Furuta inequality for the class of 2n-perinormal operators , *Global Journal of Advanced research* , 8 (4) , 1-5.

Wanjala, V. & Adhiambo, O. (2021) . On N-A-Metrically equivalent and A- Metrically equivalent operators , *International Journal of Mathematics and its Applications* , 9(2) , 97-100.

Wanjala, V. & Adhiambo, O. (2021) . On almost class (Q) and class (M,n) operators , *International Journal of Mathematics and its Applications* , 9(2) , 115-118.

Wanjala, V. & Nyongesa, A. M. (2021) . On (α,β)-class (Q) Operators , *International Journal of Mathematics and its Applications* , 9(2) , 111-113.

Wanjala, V. & Nyongesa, A. M. (2021) . On (n,m)-metrically equivalent operators , *International Journal of Mathematics and its Applications* , 9(2) , 101-104.

Wanjala, V. & Kiptoo, R . (2021) . On K^* Quasi-n-Class (Q) operators , *International Journal of Mathematics and its Applications* , 9(2) , 189-193.

Wanjala, V. & Nyongesa, A. M. (2021) . On class (Q^*) operators , *International Journal of Mathematics and its Applications* , 9(2) , 227-230.