MAASAI MARA UNIVERSITY
REGULAR UNIVERSITY EXAMINATIONS
2023/2024 ACADEMIC YEAR
THIRD YEAR SECOND SEMESTER
SCHOOL OF PURE APPLIED AND HEALTH SCIENCES
THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS

MAT 3224-1: INTRODUCTION TO MATHEMATICAL MODELING

Instructions to candidates:

Answer Question 1. And any other TWO.

All Symbols have their usual meaning

DATE: TIME:

Question 1(20 Marks)

(a) What is mathematical modeling (**1 Marks**)

(b) Explain two advantages of mathematical modeling (**4 Marks**)

(c) The population growth of a certain specie is such that the rate of growth, r, of the population is proportional to the population itself. Further, the population is not limited by any factors.

(i) Use the information above to set up a differential equation that describes the population. Use P(t) to represent the population at any time t. (**2 Marks**)

(ii) With the initial population $P(0) = P_o$, solve the differential equation in (i) above and describe your solution (**3 Marks**)

(d) The current $I(t)$ in an electric circuit is modeled by the differential equation

$$\frac{dI}{dt} = 15 - 3I$$

Given that the initial current in the circuit is $I(o) = 0$,
solve the differential equation and determine the limiting value of the current (**5 Marks**)

(e) Consider the motion of an object with mass m at the end of a vertical spring. If the spring is stretched x units from its natural position, then it exerts a force proportional to x, that is a restoring force given by $F = -kx$, where k is the spring constant. Using Newton's second law together with the force $F = -kx$, form a differential equation that describes the motion of the object and solve it. (**5 Marks**)

Question 2 (15 Marks)

One of the simplest model of population growth is described by the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{1}$$

where N=N(t) represent the population at any time t.

(i) Interpret r and K (**4 Marks**)

(ii) Find the analytic solution of the model in (1) (**7 Marks**)

(iii) What happens to the solution in (ii) above as $t \to \infty$, have a sketch to describe your answer (**4 Marks**)

Question 3 (15 Marks)

A certain infectious disease spreads in a population and is described by the model below

$$\begin{aligned}\frac{dS}{dt} &= \phi N - \beta SI - \mu S\\ \frac{dI}{dt} &= \beta SI - rI - \mu I\\ \frac{dR}{dt} &= rI - \mu R,\end{aligned}$$

where ϕ is the recruitment rate of individuals which is assumed to balance with the natural death μ in all the compartments, β is the transmission rate of the disease and r is the recovery rate.

(a) Represent the progression of individuals from one compartment to another using a schematic diagram **(2 Marks)**

(b) Show that the total population under consideration is a constant **(3 Marks)**

(c) Determine the disease free equilibrium (DFE) of the model in **(2 Marks)**

(d) Determine the basic reproduction number,R_o, of the model **(5 Marks)**

(e) With $\phi = \mu = 0.005$,$\beta = 0.4$,$r = 0.1$. Find the numerical value of R_o. Using R_o state whether there will be an epidemic or not. **(3 Marks)**

Question 4 (15 Marks)

Suppose two species X and Y are to be introduced onto an island. It is known that the two species compete, but the precise nature of their interactions is unknown. We assume that the populations $x(t)$ and $y(t)$ of X and Y, respectively, at time t are modeled by a system

$$\dot{x} = f(x, y), \tag{2}$$

$$\dot{y} = g(x, y). \tag{3}$$

In the questions below, justify your answers.

(i) Suppose $f(0, 0) = g(0, 0) = 0$; that is, $(x, y) = (0, 0)$ is an equilibrium point. What does this say about the ability of X and Y to migrate to the island? **(2 Marks)**

(ii) Suppose that a small population of just X or just Y will rapidly reproduce. What does this imply about $f_x(0, 0)$ and $g_y(0, 0)$? **(2 Marks)**

(iii) Since X and Y compete for resources, the presence of either of the species will decrease the rate of growth of the population of the other. What does this say about $f_y(0,0)$ and $g_x(0,0)$? (**2 Marks**)

(iv) Using the assumption from parts (i) through (iii), what type(s) of equilibrium point (sink, source, center, and so on) could $(0,0)$ possibly be?[*Hint*: there may be more than one possibility; if so list them all.] (**3 Marks**)

Suppose that species X reproduces very quickly if it is on the island without any Y's present, and that species Y reproduces slowly if there are no X's present. Also suppose that the growth rate of species X is decreased a relatively large amount by the presence of Y, but that species Y is indifferent to X's population.

(v) What can you say about $f_x(0,0)$ and $g_y(0,0)$? (**2 Marks**)

(vi) What can you say about $f_y(0,0)$ and $g_x(0,0)$? (**2 Marks**)

(vii) What are the possible type(s)(sink, source, center, and so on) for the equilibrium point at $(0,0)$ [*Hint*: there may be more than one possibility; if so list them all.] (**2 Mark**)