MAASAI MARA UNIVERSITY

REGULAR UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

SCHOOL OF PURE, APPLIED AND HEALTH SCIENCES

MASTER OF SCIENCE EXAMINATION

COURSE CODE: MAT 8106

COURSE TITLE: GROUP THEORY

DATE: **TIME: 3 Hours**

INSTRUCTIONS TO CANDIDATES

Answer Question **ONE** and any other **TWO** questions

*This paper consists of **THREE** printed pages. Please turn over.*

QUESTION ONE – 30 MARKS

a) State the fundamental theorem of finitely generated abelian groups. **(2 Marks)**

b) State and prove the Jordan-Holder theorem. **(4 Marks)**

c) Suppose H acts on K with an action ϕ such that the group $G = H_{\phi} \times K$. Show that $H \leq G$, $HK = G$ and $H \cap K = I$. **(6 Marks)**

d) i) Distinguish between a lower central series and an upper central series. **(2 Marks)**

ii) Show that every nilpotent group is soluble and hence by a counter example, demonstrate that the converse is not true. **(6 Marks)**

e) Define a composition series and hence find all the composition series of $\mathbb{Z}_5 \times \mathbb{Z}_5$.

(5 Marks)

f) Let n and m be relatively prime. Show that $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$. **(5 Marks)**

QUESTION TW0 – 15 MARKS

a) Define a p - group and hence show that every p - group is nilpotent. **(5 Marks)**

b) Show that a finite group G is soluble if it contains a normal subgroup K such that K and G/K are soluble. **(5 Marks)**

c) Define a soluble group and hence prove that all finite abelian groups are soluble.

(5 Marks)

QUESTION THREE – 15 MARKS

a) i) Differentiate between external direct product and internal direct product. **(2 Marks)**

ii) Prove that $D_4 = \langle (a,b) : a^4 = I = b^2, bab^{-1} = a^{-1} \rangle$ of order 8 is a splitting extension of a cyclic group of order 4 by a cyclic group of order 2. **(4 Marks)**

b) Find all abelian groups of order 504 up to isomorphism. **(4 Marks)**

c) Find the order of $(3,10,9)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$. **(5 Marks)**

QUESTION FOUR – 15 MARKS

a) List all elements of $\mathbb{Z}_3 \times \mathbb{Z}_4$. Is this group cyclic? If so determine the generator and hence deduce that $\mathbb{Z}_3 \times \mathbb{Z}_4$ is isomorphic to $\mathbb{Z}_{12}$. **(4 Marks)**

b) If m is a square free integer, then prove that every abelian group of order m is cyclic. **(3 Marks)**

c) Show that for any groups H and K, $H \times K \cong K \times H$. **(3 Marks)**

d) Prove that the external direct product of H and K is abelian if and only if H and K are abelian groups. **(5 Marks)**

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