MAASAI MARA UNIVERSITY

MAIN EXAMINATION 2023/2024 ACADEMIC YEAR

MASTER OF SCIENCE FIRST SEMESTER EXAMINATIONS

FOR

THE DEGREE OF MATER OF SCIENCE IN PURE MATHEMATICS

MAT 8103: FUNCTIONAL ANALYSIS 1

DATE: TIME:

DURATION: 3 hours

INSTRUCTIONS TO CANDIDATES

1. This paper contains **FOUR** (4) questions.
2. Answer any THREE (**3**) questions.
3. Do not forget to write your Registration Number.

QUESTION ONE (20MARKS)

a) Define a complete metric space and hence show that $\mathbb{R}^n$ is a complete metric space. **10marks**

b) Define strong and weak convergence of a sequence and hence show that the strong convergence of a sequence (x_n) implies weak convergence with the same limit. **5marks**

c) Let $a \in \mathbb{R}^3$. Define $f:\mathbb{R}^3 \to \mathbb{R}$ by $f(x) = \langle x, a \rangle$ for all $x \in \mathbb{R}^3$. Show that f is a bounded linear functional with $\| f \| = \|a\|$ **5marks**

QUESTION TWO (20MARKS)

a) Let X be a non-empty metric space. Suppose that X is complete and let $T: X \to X$ be a contraction on X. Show that T has one fixed point **10marks**

b) Define an isomorphism of a normed space and show that the dual of $\mathbb{R}^n$ is $\mathbb{R}^n$ **10marks.**

QUESTION THREE (20MARKS)

a) Define a bounded linear transformation and hence show that an integral operator defined by $Tx(t) = \int_a^b x(t)\, dx \quad \forall\, x, y \in C_{[a,b]}$ is linear and bounded **6marks**

b) Define a compact metric space and hence show that every closed subset of a compact metric space is compact **6marks**

c) Show that every bounded linear functional f on a Hilbert space H can be represented in terms of inner product namely $f(x) = \langle x, z \rangle$ where z depends on f and has norm $\|z\| = \| f \|$. **8 marks**

QUESTION FOUR (20MARKS)

a) Define
   i. Normed linear space **2marks**
   ii. Inner product space **2marks**
   iii. Weak*convergence **1mark**

b) Show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$ **3marks**

c) Determine whether the metric space (X,d) where $X=\left\{\frac{1}{n}, n\in\mathbb{N}\right\}$ is compact **2marks**

d) Let X be a non-empty complete metric space, nonmeager and $X\neq\bigcup_{k\neq1}^{\infty}A_k$.

Show that at least one A_k contains a nonempty open subset. **10marks**