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Separation axioms on function spaces defined on bitopological spaces

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Abstract

In this paper, we generalize separation axioms to the function space $p-C_{\omega}(Y,Z)$ and study how they relate to separation axioms defined on the spaces (Z,δ_i) for $i=1,2$, (Z,δ_1,δ_2), $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$. We show that the space $p-C_{\omega}(Y,Z)$ is ${}_pT_{\circ}$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if the spaces (Z,δ_1) and (Z,δ_2) are both $T_{\circ}$, T_1, T_2 and regular respectively. The space $p-C_{\omega}(Y,Z)$ is also shown to be ${}_pT_{\circ}$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if the space (Z,δ_1,δ_2) is $p-T_{\circ}$, $p-T_1$, $p-T_2$ and p-regular respectively. Finally, the space $p-C_{\omega}(Y,Z)$ is shown to be ${}_pT_{\circ}$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if and only if the spaces $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ are both T_0, T_1, T_2, and only if the spaces $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ are both regular respectively.

Keywords: bitopological space, function space, separation axiom.
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1. Introduction

The set of all continuous functions from a topological space Y to a topological space Z is denoted by $C(Y,Z)$. Several topologies have been defined on this set as seen in [3], [1] and [2]. The non empty set Y when assigned two unique topologies τ_1 and τ_2, forms a bitopological space (Y,τ_1,τ_2) (see [5]). A number of function spaces have been defined on sets of continuous functions between two bitopological spaces (Y,τ_1,τ_2) and (Z,δ_1,δ_2), examples of such function spaces include; $s-C_{\tau}(Y,Z)$, $p-C_{\omega}(Y,Z)$, $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ (see [8]).

Separation axioms allows one to separates points from points, points from closed sets and closed sets from each other using open sets. These axioms play a critical role in topology in that, apart from characterizing continuous mappings, they also provide restrictive conditions on which other topological properties and structures can be defined on a given non empty set. Studies of separation axioms on function spaces are covered in [1], [4] and [12]. Pairwise separation axioms have been introduced on bitopological spaces in [5], while in [6] and [11], comparisons have been made between separation axioms defined on the spaces

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(Y, τ_1) and (Y, τ_2), (Y, τ_1, τ_2) and $(Y, \tau_1 \vee \tau_2)$. In this paper, we generalize separation axioms to the function space $p - C_\omega(Y, Z)$, and study how they relate to separation axioms defined on topological spaces (Z, δ_i) for $i = 1, 2$, pairwise separation axioms defined on bitopological space (Z, δ_i, δ_2), as well as separation axioms defined on function spaces $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$.

2. Preliminaries

The following definitions are considered in this paper.

Definition 2.1. A function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$, is said to be pairwise continuous (p-continuous) or $\tau_1 - \delta_1$ and $\tau_2 - \delta_2$ continuous, if the induced functions $f : (Y, \tau_1) \longrightarrow (Z, \delta_1)$ and $f : (Y, \tau_2) \longrightarrow (Z, \delta_2)$ are both continuous (see [10]).

Definition 2.2. The collection $S((U, V), (A, B))_p = \{f \in p - C(Y, Z) : f(U) \subset V \text{ and } f(A) \subset B\}$ of sets, for U open in τ_1, V open in δ_1, A open in τ_2 and B open in δ_2, forms the subbasis for the open-open topology ω on $p - C(Y, Z)$ (the set of all pairwise continuous functions). If U and A are compact subsets of Y, then $S((U, V), (A, B))_p$ forms the subbasis for compact open topology. The set of all pairwise continuous functions endowed with topology ω is denoted by $p - C_\omega(Y, Z)$ (see [9]) .

Definition 2.3. The space (Y, τ_1, τ_2) is said to be pairwise $T_\circ$ $(p - T_\circ)$, if for each pair of distinct points of Y, there is a τ_1 open set or τ_2 open set containing one of the points, but not the other (see [7]).

Definition 2.4. The space (Y, τ_1, τ_2) is said to be pairwise T_1 $(p - T_1)$, if for each pair of distinct points $x, y \in Y$, there is a τ_1 open set U and a τ_2 open set V, such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$ (see [11]).

Definition 2.5. The space (Y, τ_1, τ_2) is said to be pairwise T_2 $(p - T_2)$, if for two distinct points $x, y \in Y$, there is a τ_1 open set U and τ_2 open set V, such that $x \in U$, $y \in V$ and $U \cap V = \phi$ (see [5]).

Definition 2.6. In the space (Y, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2, if for each $y \in Y$ and τ_1 closed set F such that $y \notin F$, there exist τ_1 open set U and τ_2 open set V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$. The space (Y, τ_1, τ_2) is said to be pairwise regular (p-regular), if it is both τ_1 regular with respect to τ_2 and τ_2 regular with respect to τ_1 (see [5]).

Let (Y, τ_1, τ_2) and (Z, δ_1, δ_2) be bitopological spaces, and let U_1 and U_2 be open sets in τ_1, V_1 and V_2 be open sets in δ_1, A_1 and A_2 be open sets in τ_2 and B_1 and B_2 be open sets in δ_2. Let $_pT_i$ for $i = 0, 1, 2$ and $_p$regular, denote separation axioms defined on $p - C_\omega(Y, Z)$, to differentiate them from pairwise separation axioms defined on bitopological space (Y, τ_1, τ_2).

The following definitions are introduced.

Definition 2.7. A function space $p - C_\omega(Y, Z)$ is said to be a $_pT_\circ$-space, if for any two distinct functions f and g in $p - C(Y, Z)$, there exist an open set $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) : f(U_1) \subset V_1 \text{ and } f(A_1) \subset B_1\}$ neighborhood of f not containing g, or $S((U_2, V_2)(A_2, B_2))_p = \{g \in p - C(Y, Z) : g(U_2) \subset V_2 \text{ and } g(A_2) \subset B_2\}$ neighborhood of g not containing f.

Definition 2.8. A function space $p - C_\omega(Y, Z)$ is said to be a $_pT_1$-space, if for any two distinct functions f and g in $p - C(Y, Z)$, there exist open sets $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) : f(U_1) \subset V_1 \text{ and } f(A_1) \subset B_1\}$ neighborhood of f not containing g, and $S((U_2, V_2)(A_2, B_2))_p = \{g \in p - C(Y, Z) : g(U_2) \subset V_2 \text{ and } g(A_2) \subset B_2\}$ neighborhood of g not containing f.

Definition 2.9. A function space $p - C_\omega(Y, Z)$ is said to be a $_pT_2$-space, if for any two distinct functions f and g in $p - C(Y, Z)$, there exist disjoint open sets $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) : f(U_1) \subset V_1 \text{ and } f(A_1) \subset B_1\}$ and $S((U_2, V_2)(A_2, B_2))_p = \{g \in p - C(Y, Z) : g(U_2) \subset V_2 \text{ and } g(A_2) \subset B_2\}$ neighborhoods of f and g respectively.

Definition 2.10. A function space $p - C_\omega(Y, Z)$ is said to be a $_p$regular space, if for any two distinct functions f and g in $p - C(Y, Z)$ and a closed set $\overline{S((U, V)(A, B)}$ in $p - C(Y, Z)$ such that $g \notin \overline{S(U, V)(A, B)}$, there exist disjoint open sets $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) : f(U_1) \subset V_1 \text{ and } f(A_1) \subset B_1\}$ containing $\overline{S((U, V)(A, B)}$ and $S((U_2, V_2)(A_2, B_2))_p = \{g \in p - C(Y, Z) : g(U_2) \subset V_2 \text{ and } g(A_2) \subset B_2\}$ neighborhood of g.

3. Comparison of separation axioms defined on the spaces $p - C_\omega(Y, Z)$, (Z, δ_1), (Z, δ_2) and (Z, δ_1, δ_2)

Let P denote a topological property, If both the topological spaces (Z, δ_1) and (Z, δ_2), and both the function spaces $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$ have the property P, then it will be denoted by $b - P$. In this section, we establish the relationship between ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular separation axioms defined on the function space $p - C_\omega(Y, Z)$, and $b - T_\circ$, $b - T_1$, $b - T_2$ and b-regular separation axioms defined on the topological spaces (Z, δ_1) and (Z, δ_2), as well as $p - T_\circ$, $p - T_1$, $p - T_2$ and p-regular separation axioms defined on bitopological space (Z, δ_1, δ_2). We provide proof for ${}_pT_2$ and ${}_p$regularity on $p - C_\omega(Y, Z)$ whenever (Z, δ_1) and (Z, δ_2) are $b - T_2$ and b-regular spaces, and also ${}_pT_\circ$, ${}_pT_2$ and ${}_p$regularity on $p - C_\omega(Y, Z)$, whenever (Z, δ_1, δ_2) is $p - T_0$, $p - T_2$ and p-regular space. The proofs for the other separation axioms can be done in a similar manner.

Theorem 3.1. Let (Z, δ_1) and (Z, δ_2) be $b - T_2$ spaces, then $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$, and let (Z, δ_1) and (Z, δ_2) be $b - T_2$ spaces. Then there exist disjoint open sets $U_1 \in \delta_1$ and $V_1 \in \delta_1$ and also $U_2 \in \delta_2$ and $V_2 \in \delta_2$ such that $f(y) \in U_1$ and $g(y) \in V_1$, and also $f(y) \in U_2$ and $g(y) \in V_2$ respectively. Now, the disjoint open sets $S((\{y\}, U_1)(\{y\}, U_2))_p$ and $S((\{y\}, V_1)(\{y\}, V_2))_p$ in $p - C_\omega(Y, Z)$, are neighbourhoods of f and g respectively in the space $p - C_\omega(Y, Z)$. Therefore, the space $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space. □

Theorem 3.2. Let the spaces (Z, δ_1) and (Z, δ_2) be b-regular, then $p - C_\omega(Y, Z)$ with compact open topology ω is a ${}_p$regular space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$ and let $S((U_i, V_i)(U_j, (V_j)) = \{f \in p - C(Y, Z) : f(U_i) \subset V_i \; and \; f(U_j) \subset V_j)\}$ for $U_i \in \tau_1$, $V_i \in \delta_1$, $U_j \in \tau_2$ and $V_j \in \delta_2$ for $i, j = 1, 2, 3, 4....n$ be the neighbourhood system for f. Since U_i and U_j are compact, then both $f(U_i)$ and $f(U_j)$ are also compact, and since (Z, δ_1) and (Z, δ_2) are b-regular spaces, then there exist open sets A_i and B_j in δ_1 and δ_2 respectively, for $i, j = 1, 2, 3, 4....n$, such that $f(U_i) \subset A_i$, $f(U_j) \subset B_j$, $\overline{A_i} \subset V_i$ and $\overline{B_j} \subset V_j$. This implies that $S((U_i, A_i)(U_j, B_j)) \subset S((U_i, \overline{A}_i)(U_j, \overline{B}_j)) \subset S((U_i, V_i)(U_j, V_j))$. Suppose that $\overline{S((U_i, A_i)(U_j, B_j))} \subset S((U_i, \overline{A}_i)(U_j, \overline{B}_j))$, let $g \notin S((U_i, V_i)(U_j, V_j))$, then it follows that $g \notin S((U_i, \overline{A}_i)(U_j, \overline{B}_j))$, implying further that for some point $y \in Y$, $g(y) \in \overline{A_i}^{\,c}$ and $g(y) \in \overline{B_j}^{\,c}$. Thus, $S((\{y\}, \overline{A_i}^{\,c})(\{y\}, \overline{B_j}^{\,c}))$ is a neighbourhood system for g which does not intersect $S((U_i, \overline{A}_i)(U_j, \overline{B}_j))$. Since $\overline{S((U_i, A_i)(U_j, B_j))} \subset S((U_i, \overline{A}_i)(U_j, \overline{B}_j))$, then $\overline{S((U_i, A_i)(U_j, B_j))} \subset S((U_i, V_i)(U_j, V_j))$. Therefore the sets $\bigcap_{i=1}^{n} S((\{y\}, \overline{A_i}^{\,c})(\{y\}, \overline{B_j}^{\,c}))$ and $\bigcap_{i,j=1}^{n} S((U_i, V_i)(U_j, V_j))$ are disjoint open sets containing g and $\overline{\bigcap_{i=1}^{n} S((U_i, A_i)(U_j, B_j))}$ respectively, hence $p - C_\omega(Y, Z)$ is a ${}_p$regular space. □

Theorem 3.3. Let (Z, δ_1, δ_2) be $p - T_\circ$ space, then $p - C_\omega(Y, Z)$ is a ${}_pT_\circ$ space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$, since (Z, δ_1, δ_2) is a $p - T_\circ$ space, then there exist an open set $U_1 \in \delta_1$ containing $f(y)$ but not $g(y)$ or $V_2 \in \delta_2$ containing $g(y)$ but not $f(y)$. Suppose there exist an open set $U_1 \in \delta_1$ containing $f(y)$ but not $g(y)$, then by pairwise continuity of f, we can find an open set $U_2 \in \delta_2$ also containing $f(y)$ but not $g(y)$. Suppose there exist an open set $V_2 \in \delta_2$ containing $g(y)$ but not $f(y)$, then by pairwise continuity of g, we can also find an open set $V_1 \in \delta_1$ containing $g(y)$ but not $f(y)$. Either way, there exist an open set $S((\{y\}, U_1)(\{y\}, U_2))_p$ in $p - C_\omega(Y, Z)$, neighbourhood of f not containing g, or an open set $S((\{y\}, V_1)(\{y\}, V_2))_p$ in $p - C_\omega(Y, Z)$, neighborhood of g not containing f. Therefore, the space $p - C_\omega(Y, Z)$ is a ${}_pT_\circ$ space. □

Theorem 3.4. Let (Z, δ_1, δ_2) be totally disconnected $p - T_2$ space, then $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$, since (Z, δ_1, δ_2) is a totally disconnected $p - T_2$ space, then there exist disjoint open sets $U_1 \in \delta_1$ and $V_2 \in \delta_2$ containing $f(y)$ and

$g(y)$ respectively, such that $U_1 \cup V_2 = Y$. But since f and g are both $\tau_1 - \delta_1$ and $\tau_2 - \delta_2$ continuous, it follows that there exist open sets $U_2 \in \delta_2$ containing $f(y)$ and $V_1 \in \delta_1$ containing $g(y)$. Suppose $U_2 = {V_2}^c \in \delta_2$ and $V_1 = {U_1}^c \in \delta_1$. Now, ${V_2}^c \cup {U_1}^c = (V_2 \cap U_1)^c = (\phi)^c = Y$, implying that $U_2 \cup V_1 = Y$, Now, $U_2 \cap V_1 = {V_2}^c \cap {U_1}^c = (V_2 \cup U_1)^c = Y^c = \phi$. Therefore the sets U_2 and V_1 are disjoint open sets, neighbourhoods of $f(y)$ and $g(y)$ respectively. Therefore the sets $S((\{y\}, U_1)(\{y\}, U_2))_p$ and $S((\{y\}, V_1)(\{y\}, V_2))_p$ in $p - C_\omega(Y, Z)$ are disjoint open sets, neighbourhoods of f and g respectively. Hence, $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space. □

Theorem 3.5. Let the space (Z, δ_1, δ_2) be pairwise regular, then $p - C_\omega(Y, Z)$ is a ${}_p$regular space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$ and let $S((U_i, V_i)(U_j, (V_j)) = \{f \in p - C(Y, Z) : f(U_i) \subset V_i \text{ and } f(U_j) \subset V_j)\}$ for $U_i \in \tau_1$, $V_i \in \delta_1$, $U_j \in \tau_2$ and $V_j \in \delta_2$ for $i, j = 1, 2, 3, 4....n$ be the neighbourhood system for f. Now, U_i and U_j are both compact, therefore $f(U_i)$ and $f(U_j)$ are also compact. Since (Z, δ_1, δ_2) is pairwise regular space, then δ_1 regularity with respect to δ_2 implies that there exist open sets B_j in δ_2 for $j = 1, 2, 3, 4....n$, such that $f(U_j) \subset B_j$ and $\overline{B_j} \subset V_j$. This implies that $S(U_j, B_j) \subset S(U_j, \overline{B}_j) \subset S(U_j, V_j)$. Suppose that $\overline{S(U_j, B_j)} \subset S(U_j, \overline{B}_j)$, let $g \notin S(U_j, V_j)$, then it follows that $g \notin S(U_j, \overline{B}_j)$, implying further that for some point $y \in Y$, $g(y) \in \overline{B_j}^{\,c}$. Thus, $S(\{y\}, \overline{B_j}^{\,c})$ is a neighbourhood system for g which does not intersect $S(U_j, \overline{B}_j)$. Since $\overline{S(U_j, B_j)} \subset S(U_j, \overline{B}_j)$, then $\overline{S(U_j, B_j)} \subset S(U_j, (V_j)$. Therefore $\bigcap_{j=1}^{n} S(\{y\}, \overline{B_j}^{\,c})$ and $\bigcap_{j=1}^{n} S(U_j, \overline{B}_j)$ are $\tau_2 - \delta_2$ disjoint open sets neighbourhoods of g and $\overline{\bigcap_{i=1}^{n} S(U_j, B_j)}$ respectively. Now, δ_2 regularity with respect to δ_1 implies that there exist open sets A_i in δ_1 for $i = 1, 2, 3, 4....n$, such that $f(U_i) \subset A_i$ and $\overline{A_i} \subset V_i$. This implies that $S(U_i, A_i) \subset S(U_i, \overline{A}_i) \subset S(U_i, V_i)$. Suppose that $\overline{S(U_i, A_i)} \subset S(U_i, \overline{A}_i)$, let $g \notin S(U_i, V_i)$, then it follows that $g \notin S(U_i, \overline{A}_i)$, implying further that for some point $y \in Y$, $g(y) \in \overline{A_i}^{\,c}$. Thus, $S(\{y\}, \overline{A_i}^{\,c})$ is a neighbourhood system for g which does not intersect $S(U_i, \overline{A}_i)$. Since $\overline{S(U_i, A_i)} \subset S(U_i, \overline{A}_i)$, then $\overline{S(U_i, A_i)} \subset S(U_i, V_i)$. Therefore $\bigcap_{i=1}^{n} S(\{y\}, \overline{A_i}^{\,c})$ and $\bigcap_{i=1}^{n} S(U_i, V_i)$ are $\tau_1 - \delta_1$ disjoint open sets, neighbourhoods of g and $\overline{\bigcap_{i=1}^{n} S(U_i, A_i)}$ respectively. Let $f \in \overline{S(U_i, A_i)}$ and $f \in \overline{S(U_j, B_j)}$ imply that $f \in \overline{S((U_i, A_i), (U_j, B_j))}$, then $\bigcap_{i,j=1}^{n} S((\{y\}, \overline{A_i}^{\,c})(\{y\}, \overline{B_j}^{\,c}))$ and $\bigcap_{i,j=1}^{n} S((U_i, V_i)(U_j, V_j))$ are disjoint open sets neighbourhoods of g and $\overline{\bigcap_{i,j=1}^{n} S((U_i, A_i), (U_j, B_j))}$ respectively in $p - C_\omega(Y, Z)$. Therefore $p - C_\omega(Y, Z)$ is a ${}_p$regular space. □

4. Comparison of separation axioms defined on the spaces $p - C_\omega(Y, Z)$, $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$

The relationship between ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular separation axioms defined on the function space $p - C_\omega(Y, Z)$, and $b - T_\circ$, $b - T_1$, $b - T_2$ and b-regular separation axioms defined on function spaces $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$, are established in this section. We provide proof for ${}_pT_2$ and ${}_p$regular separation axioms on $p - C_\omega(Y, Z)$ whenever (Z, δ_1) and (Z, δ_2) are $b - T_2$ and b-regular spaces, and also $b - T_2$ property on (Z, δ_1) and (Z, δ_2) whenever $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space. The proofs of the other separation axioms on the function space $p - C_\omega(Y, Z)$ can be done in a similar manner as that of ${}_pT_2$.

Theorem 4.1. The function space $p - C_\omega(Y, Z)$ is a ${}_pT_2$-space, if and only if the function spaces $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$ are $b - T_2$-spaces.

Proof. Let f and g be unique functions in $p - C_\omega(Y, Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$, and let $1 - C_\varsigma(Y, Z)$ be a T_2 space such that $S(U_1, V_1)$ and $S(U_2, V_2)$ are disjoint open sets, neighbourhoods of f and g respectively. Also, let $2 - C_\zeta(Y, Z)$ be a T_2 space such that $S(A_1, B_1)$ and $S(A_2, B_2)$ are disjoint open sets, neighbourhoods f and g respectively. Now, pairwise continuity of f and g allows us to pick $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) :$

$f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p - C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ as disjoint open sets in $p - C_\omega(Y,Z)$, containing f and g respectively. Hence $p - C_\omega(Y,Z)$ is a ${}_pT_2$ space.

Conversely, let $p - C_\omega(Y,Z)$ be a ${}_pT_2$-space and let f and g be unique functions in $p - C_\omega(Y,Z)$ such that $\forall y \in Y\ f(y) \neq g(y)$, then there exist two disjoint open sets $S((U_1,V_1)(A_1,B_1))_p = \{f \in p - C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ for U_1 open in τ_1, V_1 open in δ_1, A_1 open in τ_2 and B_1 open in δ_2, neighborhood of f, and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p - C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ for U_2 open in τ_1, V_2 open in δ_1, A_2 open in τ_2 and B_2 open in δ_2, neighborhood of g. But $S((U_1,V_1)(A_1,B_1))_p = \{f \in p - C(Y,Z) : f(U_1) \subset V_1 \textit{ and } f(A_1) \subset B_1\}$=$\{\{f \in p - C(Y,Z) : f(U_1) \subset V_1\}$ and $\{f \in p - C(Y,Z) : f(A_1) \subset B_1\}\}$. Now $\{f \in p - C(Y,Z) : f(U_1) \subset V_1\} = \{f \in 1 - C(Y,Z) : f(U_1) \subset V_1\} = S(U_1,V_1)$, and $\{f \in p - C(Y,Z) : f(A_1) \subset B_1\} = \{f \in 2 - C(Y,Z) : f(A_1) \subset B_1\} = S(A_1,B_1)$. These two sets are open and are both neighborhood of f in $1 - C_\varsigma(Y,Z)$ and $2 - C_\zeta(Y,Z)$ respectively. In a similar manner, $S(U_2,V_2)$ and $S(A_2,B_2)$ are both open set, neighborhood of g in $1 - C_\varsigma(Y,Z)$ and $2 - C_\zeta(Y,Z)$ respectively. Now, $S(U_1,V_1)$ and $S(U_2,V_2)$ in $1 - C_\varsigma(Y,Z)$ are disjoint open neighborhoods of f and g respectively. Also, $S(A_1,B_1)$ and $S(A_2,B_2)$ in $2 - C_\zeta(Y,Z)$ are disjoint open neighbourhood of f and g respectively. Therefore, $1 - C_\varsigma(Y,Z)$ and $2 - C_\zeta(Y,Z)$ are $b - T_2$ spaces. □

Theorem 4.2. The function space $p - C_\omega(Y,Z)$ is a ${}_p$regular space, if $1 - C_\varsigma(Y,Z)$ and $2 - C_\zeta(Y,Z)$ are b-regular spaces.

Proof. let f and g be unique functions in $p - C_\omega(Y,Z)$ such that $\forall y \in Y\ f(y) \neq g(y)$, and let $1 - C_\varsigma(Y,Z)$ and $2 - C_\zeta(Y,Z)$ be b-regular. Then for a closed set $\overline{S(U_1,V_1)}$ in $1 - C_\varsigma(Y,Z)$ such that $f \notin \overline{S(U_1,V_1)}$, there exist disjoint open sets $S(A_1,B_1)$ and $S(C_1,D_1)$ such that $f \in S(A_1,B_1)$ and $\overline{S(U_1,V_1)} \subset S(C_1,D_1)$. Similarly, for a closed set $\overline{S(U_2,V_2)}$ in $2 - C_\zeta(Y,Z)$ such that $f \notin \overline{S(U_2,V_2)}$, there exist disjoint open sets $S(A_2,B_2)$ and $S(C_2,D_2)$ such that $f \in S(A_2,B_2)$ and $\overline{S(U_2,V_2)} \subset S(C_2,D_2)$. Since f is pairwise continuous, we have that $f \in S((A_1,B_1)(A_2,B_2))$. Now, suppose $g \in \overline{S(U_1,V_1)} \subset S(C_1,D_1)$ and $g \in \overline{S(U_2,V_2)} \subset S(C_2,D_2)$ imply that $g \in \overline{S((U_1,V_1)(U_2,V_2))}$, then $g \in \overline{S((U_1,V_1)(U_2,V_2))} \subset S((C_1,D_1)(C_2,D_2))$. Now $\overline{S((U_1,V_1)(U_2,V_2))}$ is a closed subset of $p - C_\omega(Y,Z)$ not containing f, and $S((C_1,D_1)(C_2,D_2))$ and $S((A_1,B_1)(A_2,B_2))$ are disjoint open sets containing $\overline{S((U_1,V_1)(U_2,V_2))}$ and f respectively. Therefore $p - C_\omega(Y,Z)$ is a ${}_p$regular space. □

5. Conclusion

The function space $p - C_\omega(Y,Z)$ is a ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular space, if the topological spaces (Z,δ_1) and (Z,δ_2) are $b - T_\circ$, $b - T_1$, $b - T_2$ and b-regular spaces, and also if the bitopological space (Z,δ_1,δ_2) is $p - T_\circ$, $p - T_1$, $p - T_2$ and p-regular space. The function space $p - C_\omega(Y,Z)$ is also ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if and only if the function spaces $1 - C_\varsigma(Y,Z)$ and $2 - C_\zeta(Y,Z)$ are $b - T_\circ$, $b - T_1$ and $b - T_2$, and only if the function spaces $1 - C_\varsigma(Y,Z)$ and $2 - C_\zeta(Y,Z)$ are b-regular spaces. The set $C(Y,Z)$ can be expressed as a cartesian product $\prod_{y \in Y} Z_y$. Since the product of normal spaces need not be normal, it follows that the space $p - C_\omega(Y,Z)$ need not be normal whenever (Z,δ_1) and (Z,δ_2) are both normal spaces, and also whenever (Z,δ_1,δ_2) is pairwise normal. The results so far obtained can be extended to the space $s - C_\tau(Y,Z)$ and be used to characterize compactness in the space $s - C_\tau(Y,Z)$.

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