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Multistability Analysis of a Mathematical Model of the Interaction of *Opuntia stricta* and *Dactylopius opuntiae*

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Abstract

We develop a Mathematical model showing the main dynamical regimes of the weed *Opuntia stricta* and the insect, *Dactylopius opuntiae* interaction. We prove that under appropriate conditions a positive solution of the system is asymptotically stable, unstable or it is a periodic solution. Stable equilibria points are characterised by endemic and epidemic populations. Endemic populations are regulated by the number of cacti trees available. Epidemic populations are limited by the total number of trees because mass attack of the insects may overcome resistance of any tree.

Mathematics Subject Classification: 93A30, 92B05, 34C23

Keywords: Dynamical regimes, asymptotic behaviour of solutions, bifurcation properties of solutions

1 Introduction

The arid and semi-arid regions of East Africa have many endemic spiny and succulent plant species, some of which are *Opuntia stricta*. Opuntia are native to the Americans, see for instance [7]. In Kenya, Opuntia is considered an invasive exotic weed in Laikipia. Tens of thousands of acres of rangelands in Laikipia which support wild animals and livestock have been occupied by the weed, see for instance [9]. The invasive species is a fast-growing cactus. To get rid of it probably the natives may choose to chop it and burn. Cactus does not burn. The natives may choose to use chemicals which kill the plant selectively, well it may kill the cactus but also poison the land. The poison may, most likely find its way into the grass or trees and the animals would feed on it and the result would be unpleasant (UNEP 2012). The most environmentally friendly control method will be Biological control. Biological control involves reduction of the weed population by natural enemies and typically involves an active human role, these enemies include insects and pathogens. After years of research, specialists from the Center for Agricultural Biosciences International (CABI), have introduced a sap sucking insect, *Dactylopius opuntiae,* commonly known as cochineal to control the cactus. Cochineal posses no threat to other plant species. It eats up the cactus from inside out. The cactus slowly starts to lose its fruits and eventually dries up, see for instance [9].

We wish to consider a mathematical model that describe the relationship between *Opuntia stricta* and *Dactylopius opuntiae.* Thus, we have the following model,

$$\begin{aligned} \dot{P} &= rP\left(S - P - \frac{BH}{C+P}\right), \\ \dot{H} &= DH\left(\frac{P}{C+P} - AH\right), \end{aligned} \tag{1}$$

where the dot denotes differentiation with respect to time t. The positive constants are: r, S, A, B, C and D with r being the intrinsic growth rate of the Cacti, while D is the intrinsic growth rate of the insect Cochineal at time t. The mass of the Cacti per hectare is represented by $P := P(t)$. The constant S is the measure of fertility that can support the growth of cacti and B represent the density of Cacti consumed by the insect. The population density of the insects is represented by $H := H(t)$. The constant C, determine the growth rate of the insects as they feed on the Cacti. The number of insects that die due to natural death and other causes other than lack of natural food, cacti is represented by A. To reduce the number of parameters and group them in a meaningful way, we nondimensionalize the system in Equation (1) thus:

$$t = \frac{\hat{t}}{rC}, \quad P = Cp, \quad B = Cb, \quad D = rCd, \quad H = Ch, \quad A = \frac{a}{C}, \tag{2}$$

Using, Equation(2), we obtain:

$$\dot{p} = p\left(s - p - \frac{bh}{1+p}\right),$$

$$\dot{h} \;=\; dh\left(\frac{p}{1+p} - ah\right), \tag{3}$$

where the dot represent differentiation with respect to $\hat{t}$. The parameter d is the linear growth rate of the insects to that of the cacti and so $d \geq 1$ and $d \leq 1$ have definite ecological meaning; with the later, the cacti grow faster than the insects. If $p \leq 1$, it means that predation and allee factors are negligible in this population range, see for instance [8].

2 Long Term Behaviour of the Solutions

The long term solutions of Equation (3) are useful in understanding the long term behaviour of the system, see for instance [4, 10]. We shall check the long term behaviour of the system by studying the stability of the Equilibria points.

2.1 Equilibria Points

The equalibria points of Equation (3) are given by:

$$\begin{aligned} \dot{p} &= p\left(s - p - \frac{bh}{1+p}\right) =: f(p,h) = 0, \\ \dot{h} &= dh\left(\frac{p}{1+p} - ah\right) =: g(p,h) = 0. \end{aligned} \tag{4}$$

From Equation (4), the equilibria points are, $E_0(0,0), E_1(s,0)$, and $E_2(p_0,h_0)$ that solves the following equations,

$$\begin{aligned} h_1 &= \frac{p}{1+p} - ah = 0, \\ h_2 &= s - p - \frac{bh}{1+p} = 0, \end{aligned} \tag{5}$$

where h_1 and h_2, are the nullclines of the system in Equation(3), for $p > 0, \;\; h > 0$.

The points of equilibria vary with various parameters. The point $E_0(0,0)$, represents a situation where there is no population. This condition persists because there are no members of the population to die or reproduce. The point $E_1(s,0)$, also represents a persistent behaviour. In this case however the rate of reproduction exactly balances the mortality and so the population remains constant at this level. The point $E_2(p_0,h_0)$ represents co-existence of the cacti and the insect.

2.2 Linear Stability Analysis

Stability is determined by what happens around an equilibrium point, see for instance, [1, 5]. Stable equilibria points are important,they represent fundamental features of the system. They are useful for making predictions about the system because lots of solutions eventually settle down near the stable equilibria solutions, see for instance [8]. Consider Equation (4), we shall check the stability of the points using the community matrix:

$$J := \begin{pmatrix} f_p & f_h \\ g_p & g_h \end{pmatrix}_{|E_k} \tag{6}$$

where $k =\in \{0, 1, 2\}$.

The community matrix J, has eigenvalues λ given by,

$$\lambda^2 - \tau\lambda + \delta = 0, \tag{7}$$

where, τ and δ are the trace and the determinant of J respectively. The necessary and sufficient conditions for linear stability are, $\tau < 0, \quad \delta > 0$. The equilibrium point $E_0 = (0, 0)$ is stable when, $s < 0$ and the equilibrium point $E_1 = (s, 0)$ is stable when, $s + 1 < -d$. The solution $E_n = (p_0, h_0)$, with p_0 and h_0, nonzero are given by the intersection of the nullclines.

We shall study these equilibria points using various propositions, that we now state.

Proposition 1. *For $f_h(p_0, h_0) < 0$, and $g_h(p_0, h_0) > 0$, there is an s^*, with $s* < p_0 < s, \quad h_0 < \frac{1}{a}$, such that, the equilibrium point, (p_0, h_0) is a focus and it is asymptotically stable.*

Proof. Proposition 1

Let

$$J(p, h) := \begin{pmatrix} f_p & f_h \\ g_p & g_h \end{pmatrix} \tag{8}$$

where,

$$f_h = \frac{\partial f}{\partial h}, \quad f_p = \frac{\partial f}{\partial p}, \quad g_h = \frac{\partial g}{\partial h} \quad \textit{and} \quad g_p = \frac{\partial g}{\partial p}, \tag{9}$$

Let $\delta := f_p g_h - g_p f_h$, the determinant of $J(p_0, h_0)$, and $\tau := f_p + g_h$, the trace of $J(p_0, h_0)$. For stability we have to demonstrate that $\delta > 0$ and $\tau < 0$. From Equation(4), $f_h(p_0, h_0) < 0$, and $g_h(p_0, h_0) < 0$, it therefore, follows by the Implicit Function Theorem, that there exists functions $\varphi(p)$ and $\phi(p)$ such that;

$$f(p, \varphi(p)) = 0, \tag{10}$$

$$g(p, \phi(p)) = 0, \tag{11}$$

in the neighbourhood of (p_0, h_0), the functions $\varphi(p)$ and $\phi(p)$ are as smooth as f and g respectively. Differentiating (10), (11) with respect to p in the neighbourhood of p_0, we get;

$$f_p(p, \varphi(p)) + f_h(p, \varphi(p))\varphi' = 0, \tag{12}$$

that gives, $\varphi' = \frac{-f_p}{f_h}$;

$$g_p(p, \phi(p)) + g_h(p, \phi(p))\phi' = 0, \tag{13}$$

where the prime represent differentiaton with respect to p yields,

$$\delta = -f_h g_h \varphi' - \phi'. \tag{14}$$

If, $0 < \phi' < \varphi'$, the equilibrium point (p_0, h_0), is asymptotically stable if, $\varphi' > 0$, and $\phi' > 0$.

From Equation (13) and Equation (14),

$$\tau = -f_h\varphi' - \frac{g_p}{\phi'}.$$

From Equation (5) and (6), $f_h < 0$ and $g_p > 0$. if, $\varphi' < 0$ and $\phi' > 0$, it follows that, $\tau < 0$. For, $\varphi(p_0, h_0) = 0$ and $\phi(p_0, h_0) = 0$ at the equilibrium point (p_0, h_0), where $\varphi(p; h) = s - p - \frac{bh}{1+p}$, and $\phi(p; h) = \frac{p}{1+p} - ah$, then,

$$\varphi_p = -1 + \frac{bh}{(1+p)^2}, \quad \varphi_h = \frac{-b}{(1+p)}, \tag{15}$$

$$\phi_p = \frac{1}{(1+p)^2}, \quad \phi_h = -a. \tag{16}$$

Thus δ in Equation(14) becomes:

$$\delta = dph(\varphi_p\phi_h - \varphi_h\phi_p). \tag{17}$$

Using Equations (15), (16) and (18), we get; $-a(\frac{bh}{(1+p)^2} - 1) > 0$, hence,

$$\tau = -p + \frac{bph}{(1+p)^2}.adh, \tag{18}$$

$\tau < 0$, whenever, $-1 + \frac{bh}{(1+p)^2} < 0$, and so, $bh < (1+p)^2$. Consider Equation (4), there is an equilibrium point when,

$$p(s - p - \frac{bh}{1+p}) = 0, \frac{bh}{1+p} = s - p, \quad (s-p)(1+p) < (1+p)^2,$$

The equilibrium point is:

$$p = \frac{s-1}{2}, \quad h = \frac{(s+1)^2}{4b},$$

from Equation (5),

$$h = \frac{p}{a(1+p)}, \to \frac{1}{a} \quad as \quad p \ \to \infty.$$

The population value, $\frac{s-1}{2}$ is an approximate threshold value. There is therefore an $s^* = \frac{s-1}{2}$, such that,

$$s^* < p_0 < s, \frac{(s+1)^2}{4b} < h_0 < \frac{1}{a},$$

where the equilibrium point (p_0, h_0), is asymptotically stable hence the proof.

□

Propostion 2 *Let $f_h(p_0, h_0) < 0$ and $g_p(p_0, h_0) > 0$, then there is an s^* such that whenever, $(\frac{s-d-1}{2}) < s^* < s, \quad 0 < h_0 < \frac{1}{a}$, there is a periodic solution.*

Proof. From the characteristic Equation (10),and the community matrix in Equation(7),$\delta := f_p g_h - g_p f_h$ and $\tau := f_p + g_h$.

From Equation (15) and (16), $\delta = -f_h g_h(-\phi' + \varphi')$, this shows that,$\delta > 0$, when φ' ¡ ϕ and $\tau = f_h\varphi' - \frac{g_p}{\phi'}$. For $\tau = 0$, we rquire that, $\varphi' > 0$, since $g_p > 0$, $f_h < 0$, and $\phi' > 0$. There is a periodic solution when, $\quad 0 < \phi' < \varphi'$. We now find s^*,

From Equation (4), there is an equilibrium point when, $bh = (1+p)(s-p)$,

$$\tau = -p + \frac{bhp}{(1+p)^2} - adh,$$

at equilibrium point, and

$$ah = \frac{p}{(1+p)}.$$

$\tau = 0,$ when,

$$-1 + \frac{bh}{(1+p)^2} = \frac{d}{(1+p)},$$

that upon using $bh = (s-p)(1+p)$, we obtain

$$(p^*, h^*) = \left(\frac{s-d-1}{2}, \frac{p^*}{a(1+p^*)}\right),$$

as an equilibrium point.

There is an $s^* = \frac{(s-1)}{2}$, such that, whenever,

$$\frac{s-d-1}{2} < s^* < s, \quad 0 < h_0 < \frac{1}{a},$$

there is a periodic solution, hence the proof. □

Proposition 3. *For $f_p(p_0, h_0) < 0$ and $g_p(p_0, h_0) > 0$, there is an s^* such that whenever;*

$$\frac{s-d-1}{a(s-d+1)} < s^* < s, \quad 0 < h_0 < \frac{1}{a}, \quad \frac{s-a-1}{2} < s^* < s,$$

the equilibrium point (p_0, h_0) is unstable.

Proof. We see that, $\delta > 0$, when $\phi' < \varphi'$, $\tau > 0$, implies that; $\varphi'\phi' > \frac{-g_p}{f_h}$. But $g_p > 0$, $f_h < 0$, this indicates that, $\frac{-g_p}{f_h} > 0$. The equilibrium point (p_0, h_0) is therefore unstable whenever $0 < \phi' < \varphi' < \varphi'$. Now, $\frac{s-2}{2} < p$. Consider Equation (4), then $\tau = -p + \frac{bhp}{(1+p)^2} - adh$, from Equation,(5) $ah = \frac{p}{(1+p)}$, $-1 + \frac{bh}{(1+p)^2} - \frac{d}{(1+p)} > 0$, $-1 + \frac{bh}{(1+p)^2} - \frac{d}{(1+p)} > 0$,

$$-(1+p)^2 + bh - d(1+p) > 0,$$

at equilibrium point, $bh = (s-p)(1+p)$, $-(1+p)^2 + (s-p)(1+p) - d(1+p) > 0$, $-2p > 1 - s + d$, $p < \frac{s-d-1}{2}$,

when,

$$p^* = \frac{s-d-1}{2}, \quad h^* = \frac{s-d-1}{a(s-d+1)}.$$

Consider Equation (5),

$$h_0 = \frac{1}{a}\frac{p}{(1+p)},$$

as

$$p \to \infty, \quad h \to \frac{1}{a}.$$

There is an $s^* = \frac{s-1}{2}$ such that, the equilibrium point (p_0, h_0), is unstable whenever,

$$\frac{s-d-1}{2} < s^* < s, \quad \frac{s-d-1}{a(s-d+1)} < h_0, \quad 0 < h_0 < \frac{1}{a},$$

hence the proof. □

3 Conclusion

On the strength of analysis, the result have shown that, taking into account the rate of reproduction of the cacti and insects, there are two different scenario of equilibria points that are significant. The Equilibrium point $E_0(0,0)$, represent a situation where there are no members of the cacti species to reproduce. If there is an introduction of the cacti in a given environment, the cacti population increases due to natural growth. Whereas, if the cacti population is diminishing due to environmental conditions or because it is being fed on by the insects, the condition is unstable. The Equilibrium point $E_1(s, 0)$, the cacti population is maintained at the carrying capacity with the absence of the insects, hence it is a stable equilibrium condition. A Biological control method is suitable, when the density of the weed is small, see for instance [6].

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