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ON CHARACTERIZATION OF u - IDEALS DETERMINED BY SEQUENCES

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ABSTRACT

The area of ideals is important in the study of Analysis, algebra, Geometry and Computer science. The various types of ideals have been studied, for example m ideals and h ideals. The m ideals defined on real Banach spaces are referred to as u - ideals. The natural examples of u - ideals with respect to their biduals, are order continuous Banach lattices. Using the approximation property, we shall study properties of u - ideals and their characterization. We define the set of compact operators $K(X)$ on X to be u - ideals given that X is a separable reflexive Banach space with approximation property if and only if there is a sequence (T_n) of finite rank of operators with $\lim_{n\to\infty}\|I-2T_n\|=1$ and $\lim_{n\to\infty}T_n x = x$. We shall show that u -ideals containing no copies of sequences ℓ_1 are strict u - ideals.

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INTRODUCTION

The notion of ideals was first introduced by Alfsen and Effros [4] in the early 1970's. They defined a closed subspace X of Banach Space Y to be an ideal in Y if the orthogonal complement of X in the dual of Y is the kernel of a norm one projection. That is, $\beta : Y^* \to Y^*$ such that $X^{\perp} = \{ y^* \in Y^* : y^*(x) = 0, \forall x \in X \} = Ker\beta.$ A simple example is that X is always an ideal in X^{**}. Let $Q : X^{***} \to X^{***}$ be the identity. Then, clearly Q is a projection and $KerQ = \{0\}$. Now $X \subseteq X^{**}$ and so $X^{\perp} \subseteq X^* \subseteq X^{***}$.However, $X^{\perp} = \{ x^* \in X^* \mid x^*(x) = 0, \forall x \in X \} = \{0\}$.Therefore, $KerQ = X^{\perp}$. This shows that X is always an ideal in X^{**}. Since then scholars have studied various types of ideals and their properties. They have borrowed a lot from algebra since ideals are known to have absorbing properties. For instance an ideal I of a ring R which is an additive subgroup and is such that for all $x \in R$ and $y \in I$, $x\,y \in I$. The m -ideals defined on a real Banach space are called u -ideals whereas on a complex Banach space is called h -ideals. Let X be a subspace of a Banach space Y . We will say that X is an m -summand if it is the range of a contractive projection and that X is an ideal in Y if $X^{\perp}$ is the kernel of a contractive projection on Y^* . Godefroy, Kalton and Saphar [3] defined u -ideals as the generalizations of m -ideals. The subspace $K(X,Y)$ is an ideal in $L(X,Y)$ if $K(X,Y)$ is the kernel of a contractive projection β in $L(X,Y)^*$. That is, $\beta : Y^* \to Y^*$ such that $X^{\perp} = \{ y^* \in Y^* : y^*(x) = 0 \ \ \forall \ x \in X \}$. Moreover, $K(X,Y)$ is a u -ideal in $(L(X,Y),\|.\|)$ if $\|I-2\beta\|=1$. The natural examples of u -ideals with respect to their biduals, are order-continuous Banach lattices.

In this paper we fill a few gaps in u -ideals determined by sequence spaces ℓ_1 , ℓ_∞, c_0. We show that if X is a separable u -ideal containing no copies of ℓ_1 then, it is a strict u -ideal. In section 2 we discuss u -ideals and their characterization. In section 3 we characterize strict u -ideals determined by sequence space ℓ_1.

Remark 1.1: The sequence spaces ℓ_1 and ℓ_∞ can never be strict u -ideals in their biduals since dual spaces are 1-complemented in their biduals [5].

2.0 u -IDEALS

We say that a closed subspace X of Y is a u -summand if there is a subspace Z (the u - complement of X) so that $X \oplus Z = Y$ and if $x \in X$, $z \in Z$ then $\|x+z\| = \|x-z\|$. If X is a u - summand then the induced projection $P : Y \to X$ with $P(Y) = X$ and $KerP = Z$ satisfies $\|I - 2p\| = 1$.

Lemma 2.1 : Suppose X is a closed subspace of Y. Then there is at most one projection P of Y onto X satisfying $\|I - 2p\| = 1$.

Proof: Suppose P and Q are two projections such that $\|I-2P\| = \|I-2Q\| = 1$. Then

$$(I-2p)(I-2Q) = (I-2Q) - 2P(I-2Q) = I - 2Q - 2P + 4PQ$$

Now, since $Q(Y) = X$, we have $(PQ)y = P(Qy) = Qy$, where $y \in Y$ and $Qy \in X$.

Therefore

$$(I-2p)(I-2Q) = I - 2Q - 2P + 4Q = I + 2Q - 2P = I + 2(Q-P).$$

Thus we have

$$((I-2P)(I-2Q))^2 = (I + 2(Q-P))(I + 2(P-Q))$$
$$= I + 2(Q-p) + 2(Q-P)(I + 2(Q-P))$$
$$= I + 2(Q-P) + 2(Q-P) + 4(Q-P)$$
$$= I + 4(Q-P) + 4(Q^2 - PQ - QP + P^2)$$
$$= \text{I}+2.2(Q-P).$$

$$((I-2P)(I-2Q))^3 = (I + 4(Q-P))(I + 2(Q-p))$$
$$= \text{I}+2(Q-P) + 4(Q-P) + 8(Q-P)^2$$
$$= \text{I}+2.3(Q-P) \text{ etc.}$$

In general $((I-2P)(I-2Q))^n = I + 2n(Q-p)$.

Since

$$\|I - 2n(Q-P)\| = \|I - 2n(P-Q)\| \geq |1 - 2n\|P-Q\|| \xrightarrow[\infty]{n} \infty \text{ if } \|P-Q\| \neq 0$$

and $\|((I-2P)(I-2Q))^n\| \leq \|I-2p\|^n \|I-2Q\|^n = 1$,

We have a contradiction, unless $P = Q$.

Lemma 2.2: If A is a u -ideal in B then A is a u -summand if and only if W is weak* -closed.

Proof: Clearly if W is weak*-closed then X is weak *-continuous and so $X = Y^*$ where $\|I - 2Y\| = 1$ and $Y(B) = A$. Conversely, suppose A is a u -summand and let Y be a projection onto A with $\|I - 2Y\| = 1$. Then $I - Y^*$ has a range $A^\perp$ and so $I - Y^* = I - X$ by Lemma 2.1. Hence X is weak *-continuous.

Proposition 2.1: Let X be a closed subspace of a Banach space Y . If $K(Z,X)$ is a u -ideal in $K(Z,Y)$ for some Banach space $Z \neq \{0\}$, then X is u -ideal in Y .

Proof: Suppose $\overline{K(Z,X)}$ is an ideal in $\overline{K(Z,Y)}$. Let E be a finite dimensional subspace of Y . Let $z \in Z$ and $z^* \in Z^*$ be such that $\|z\| = \|z^*\| = z^*(z) = 1$. Denote $T = \{z^* \otimes y : y \in E\} \subseteq K(Z,Y)$. Let $\varepsilon > 0$ and let $V : T \to \overline{K(Z,X)}$ be an operator such that $\|V\| \leq 1 + \varepsilon$ and $V(S) = S$ for all $s \in T \cap K(Z,X)$. Now define a map $U : E \to X$ by $U_y = (V(z^* \otimes y))z$. Then U “locally 1-complements” X in Y by local formulations of u - ideals [1, Lemma 2.9].

3. STRICT u -IDEALS

In this section we consider strict u -ideals, that is, the Banach space X which are strict u -ideals in their biduals X^{**} . It has already been show that Banach spaces containing copies of ℓ_1 are not strict u -ideals $[3, Theorem\, 5.1]$. We show that separable Banach spaces containing no copies of ℓ_1 are strict u - ideals. A Banach space X is said to be a strict u -ideal in its bidual when the canonical decomposition $X^{***} = X^* \oplus X^\perp$ is unconditional. In other words for X to be a strict u -ideal the u -complement of $X^\perp$ must be norming, that is, the range V of the induced projection on X^{***} is a norming subspace of X^*.

Remark 3.1: The sequence space ℓ_1 is a u -ideal since it is a u -summand in ℓ_1^{**}. It is therefore not a strict u -ideal (Lemma 2.1).

Proposition 3.1: Let X be a Banach space containing no copy of ℓ_1. If X is a strict u -ideal in X^{**} and $K(Z,X)$ is an Ideal in $K(Z,X^{**})$ for a reflexive Banach space Z, then $K(Z,X)$ is a strict u -ideal in $K(Z,X^{**})$.

Proof: Let $\lambda : X^{***} \to X^{***}$ be the projection from the definition of a strict u -ideal and let Q denote the ideal projection on $K(Z,X^{**})^*$. It follows that X^* does not contain any proper norming closed subspace [3, proposition 5.2]. But then X has the unique extension property thus $K(Z,X)$ is a u -ideal in $K(Z,X^{**})$. However Q is the desired u -ideal projection and $Q(x^{***} \otimes z) = (\lambda x^{***}) \otimes z$ for $x^{***} \in X^{***}$, $z \in Z$. In view of this equality the range of Q contains the functionals $x^{***} \otimes z$ with $x^{***} \in ran\lambda$ and $z \in Z$. But this functionals give the norm of any $V \in K(Z,X^{**})$ by $\left(\|V\| = \sup\{|x^{***}(Vz)| : x^{***} \in A_{ran\lambda}, z \in Az\}\right)$ because the ran λ is a norming subspace for X^{**} in X^{***} in fact $ran\lambda = X^*$ (cf. [3]).

We now characterize the u -ideals determined by the sequence space ℓ_1.

Remark 3.2: A separable Banach space containing ℓ_1

cannot be a strict u-ideal in its bidual [5].

Theorem 3.1: Let A be a u-ideal. The following are equivalent:

i) A is a strict u-ideal.

ii) A^* is a u-ideal.

iii) $\|I-2p\|=1$.

iv) Every separable subspace of A has separable dual.

v) A contains no copy of ℓ_1.

Proof. (i) $\rightarrow(ii)$ This is clear since A is a separable Banach space. In this case the operator $V: A^{**} \rightarrow A^{**}$ is an isometry. Since V is hermitian it follows that $V(A)$ = $V^2(A)$ and so V is invertible on $V(A)$. This implies that V is surjective and so its spectrum is contained in the unit circle. Since its hermitian $\delta(V) \subset \{\pm 1\}$. However $\|I-2V\|=1$ and so the spectrum of V reduces to {1}. Hence the spectrum of $V-I=0$ from Sinclair theorem $[2]$ and N=F so that A^* is u-summand in A^{***}.

(ii) $\rightarrow(iii)$ Let A^* contain no copy of c_0. Then since it is a dual space, ℓ_∞ embeds into A^* and so has the property (u); which is not true. Therefore A^* is a u-summand in A^{***}. Let F: $A^{***} \rightarrow A^*$ be a hermitian projection. Let N: $X^{***} \rightarrow V$ be the hermitian projection associated with the fact that A is a u-ideal. Then $2(FN-NF)$ is hermitian. Note that since F is also a norm one projection onto A^* and so FN is a hermitian on A^*. Hence $I_{A^*}-FN$ is a hermitian implying that $I_{A^*}-FN$ =0 on A^* and thus FN is another contractive projection onto A^*. Hence NF is a contractive projection. Thus A^*=V and F=P.

(iii) $\rightarrow(iv)$ Let A be a separable space for which A^* is separable then by [3, Theorem 2.8] the hermitian condition holds.

(iv) $\rightarrow(v)$ It is clear that A contains no copy of ℓ_1 since it has a separable dual.

(v) $\rightarrow(i)$ V is an identity on X^{**} and so $F=P$ and A is a strict u-ideal.

Proposition 3.1: Assume that X is non-reflexive. If X is a strict u-ideal in its bidual then every subspace of X contains no copy of ℓ_1.

Proof: Since V is norming the associated operator $T: X^{**} \rightarrow X^{**}$ is an isometry. If X contains a copy of ℓ_1 then, there exists $x^{**} \in X^{**}$ with $\|x^{**}\|=1$ and such that $\|x^{**}+x\|=\|x^{**}-x\|$ for all $x \in X$. If $\|I-P\|=k$ then we can find a net (x_d) in X, converging weak* to Tx^{**}, with $\limsup\|Tx^{**}-x_d\| \leq k$. Since T is an isometry and $\limsup\|x^{**}-x_d\| \leq k$, Therefore $\limsup\|x^{**}+x_d\|=\limsup\|Tx^{**}+x_d\| \leq k$. However, $\limsup\|Tx^{**}+x_d\| \geq 2$. It is clear that every subspace of a strict u-ideal in its bidual does not contain ℓ_1.

Proposition 3.2: Let X be a Banach space containing no copy of ℓ_1. The following statements are equivalent:

(i) X is a strict u-ideal.

(ii) Every separable closed subspace Y of X and every element in the bidual of Y satisfy the hermitian condition.

Proof: Assuming that X is separable. We will show that X^* is separable. Let V be a closed norming subspace of X. Then if $x^{**} \in V^{\perp}$ we have $\|x^{**}-x\| \geq\|x\|$ for all $x \in X$. In particular $\inf_{x \in s_x}\|x^{**}-2x\| \geq 2$. Therefore $V=X^*$ and since X has no proper norming subspaces it follows that X^* is separable. Let there be a sequence (x_n) converging weak* to x^{**} so that $\lim\|x^{**}-2x_n\|=1$. By density argument this holds for all $x^{**} \in s_{X^{**}}$ and which shows that $\|I-2p\|=1$. If X is nonseparable then, every separable subspace Y satisfies u-constant of Y to be 1. This implies that the u-constant of X is 1 and hence X is a strict u-ideal in X^{**} containing no copy of ℓ_1.

Proposition 3.3: Let A be a Banach space containing no copy of ℓ_1. Show that

(i) A^* has an approximating sequence (a_n) and A is a strict u-ideal iff A has an approximating sequence (a_n) .

(ii) A and A^* have an approximating sequence (a_n) iff A has an approximating sequence (a_n) .

Proof: Let (a_n) be an unconditional approximating sequence for A^*. Then since A^* contains no copy of c_0 [3, Theorem3.5] there is a projection $Q: A^{***} \rightarrow A^*$ by $Qx^{***}=\lim a_n^{**} x^{***}$. It follows that $\|I-2p\|=1$. However, if A is a strict u-ideal then $\|I-2p\|=1$ and by Lemma 2.1, Q=P. Now let $a_n^*: A^{**} \rightarrow A^{**}$. Let $c: A^{**} \rightarrow A^{**}/A$ be the quotient map, and let $J: A \rightarrow A^{**}$ be the canonical embending. Let $H_n=c a_n^{**} J: A \rightarrow A^{**}$. Then H_n^* : $A^{\perp} \rightarrow A^*$ and coincides with a_n^{**}. Thus H_n^* converges to zero for the strong operator topology implying that H_n converges to zero for a weak topology on $K\left(A, A^{**}/A\right)$. Therefore by approximating properties $\lim_{n \rightarrow \infty} a_n=a$. In (ii) A has an approximating sequence (a_n) such that (a_n^*) is an approximating sequence for A^* and such that $\lim_{n\to\infty}\|I-2a_n\|=1$. Then H_n - A_n coverges weakly to zero in $K(A)$ and so there is an approximating sequence of convex combinations R_n of H_n such that $\lim_{n\to\infty}\|I-2R_n\|$=1.

***Open questions**:*

(i) Is a Banach space X a u-ideal in X^{**} ?.

(ii) If the dual of X is a u-summand in X^{**},does it imply that it is a strict u-ideal ?.

(iii) Let X be a separable reflexive Banach space. Can we show that $K(X)$ is a u-ideal in $\ell^w(X)$ iff X has an approximating sequence?.

CONCLUSION

We have shown that u-ideals containing no copies of sequences ℓ_1 are strict u - ideals.

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