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Heredity of Lower Separation Axioms on Function Spaces

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Abstract

The set of continuous functions from topological space Y to topological space Z endowed with a topology forms the function space. For A subset of Y, the set of continuous functions from the space A to the space Z forms the underlying function space with an induced topology. The function space has properties of topological space dependent on the properties of the space Z, such as the T_0, T_1, T_2 and T_3 separation axioms. In this paper, we show that the underlying function space inherits the T_0, T_1, T_2 and T_3 separation axioms from the function space, and that these separation axioms are hereditary on function spaces.

Keywords

Function Space; Underlying Function Space; Hereditary Properties

1. Introduction

The set of continuous functions from the space Y to the space Z is denoted by $C(Y,Z)$. The set open topology τ defined on the set $C(Y,Z)$ generated by the sets of the form $F(U,V)=\{f\in C(Y,Z): f(U)\subset V\}$, where the sets U and V ranges over the class $\mathbb{C}$ of compact subsets of Y and Ω_Z class of open subsets of Z respectively, is called the compact open topology. The sets of the form $F(U,V)$ forms subbases for the compact open topology τ on $C(Y,Z)$ (see [1]). The set open topology τ defined on the set $C(Y,Z)$ generated by the subbases $S(y,U)=\{f\in C(Y,Z), f(y)\in U\}$ where $y\in Y$ and $U\in\Omega_Z$ is called point open topology (see [2]).

Let $A=\bigcap_i^n U_i$ for $\{U_i: i=1,2,3,\cdots,n\}$ family of non-empty open subsets of Y. The set $C(A,Z)$ consist

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of continuous functions of the form $f \circ i = f|_A$ where $i : A \to Y$ is an inclusion mapping (see [3]).

Let the topological space Z be a T_i-space for $i = 0,1,2,3$, then the function space $C_\tau(Y,Z)$ with compact open topology τ inherits the T_i-separation axioms for $i = 0,1,2,3$ (see [4] and [5]).

Definition 1.1 *For $A \subset Y$, the sets of the form* $C(A,Z) \cap S(y,V) = \{ f \in C(Y,Z) : f(\{y\} \cap A) \in V \} = \{ f \in C(Y,Z) : f|_A(y) \in V \} = S(y,V) \forall y \in A$ *as defined in* [3], *forms the subbases for point open topology on the set* $C(A,Z)$.

Definition 1.2 *The sets of the form* $C(A,Z) \cap F(U,V) = \{ f \in C(Y,Z) : f(A \cap U) \subset V \} = \{ f \in C(Y,Z) : f|_A(U) \subset V \} = F(U,V)$ *where U is open in A, $U \in \mathcal{C}$ and $V \in \Omega_Z$, defines the subbases for the set open topology on the set $C(A,Z)$* (see [3]). *This topology is referred to as open-open topology* (see [6]). *If U is compact, then $F(U,V)$ defines the subbases for the compact open topology on the set $C(A,Z)$.*

The point open topology and the compact open topology are also open-open topologies. The set $C(A,Z)$ endowed with set open topology ζ is written as $C_\zeta(A,Z)$ and is referred to as the underlying function space of the space $C_\tau(Y,Z)$ (see [3]).

Definition 1.3 *Let $U_\circ$ and $V_\circ$ be open subsets of Y and Z respectively. The set $C(U_\circ,V_\circ)$ forms the subspace of the function space $C_\tau(Y,Z)$ with the induced topology ϱ generated by the subbases* $C(U_\circ,V_\circ) \cap F(U,V) = \{ f \in C(Y,Z) : f(U_\circ \cap U) \subset (V_\circ \cap V) \} = \{ f \in C(Y,Z) : f(U) \subset V \} = F(U,V)$ (see [7]).

The following lemma and theorem are important for our consideration.

Lemma 1.4 *In a regular space, if F is compact, U an open subset of a regular space and $F \subset U$, then for some open set V, $F \subset U$ and $\overline{V} \subset U$.*

From the above lemma, the following inference is made. Let $K_i \in \mathrm{C}$ where C is a class of compact subsets of Y and $U_i \in \Omega_Z$. Then for the space $C_\tau(Y,Z)$ with compact open topology τ, $f(K_i)$ is a compact subset of U_i. Since Z is a regular space, there exist open sets $V_i \in \Omega_Z$, such that $f(K_i) \subset V_i$ and $\overline{V_i} \subset U_i$. This implies that $F(K_i,V_i) \subset F(K_i,\overline{V_i}) \subset F(K_i,U_i)$, in which the assertion $\overline{F(K_i,V_i)} \subset F(K_i,\overline{V_i})$ can be made (see [5]).

Theorem 1.5 *The function $\sigma : C_\zeta(A,Z) \to C_\varrho(U_\circ,V_\circ)$ defined by $\sigma(f|_A) = f$ is a homeomorphism* (see [7]).

2. Lower Separation Axioms on the Underlying Function Space $C_\zeta(A,Z)$

In this section, we show that the underlying function space $C_\zeta(A,Z)$ inherits the T_i-separation axioms for $i = 0,1,2,3$ from the space $C_\tau(Y,Z)$. Topologies τ and ζ are both compact open.

Theorem 2.1 *Let the function space $C_\tau(Y,Z)$ be a $T_\circ$ space. The function space $C_\zeta(A,Z)$ for $A \subset Y$ is a $T_\circ$ space.*

Proof. Let $f, g \in C_\tau(Y,Z)$ be distinct maps such that $\forall y \in Y$, $f(y) \neq g(y)$. Then $\forall y \in A$, $f|_A(y) \neq g|_A(y)$. For the open set $S(y,V)$ containing f but not g in $C_\tau(Y,Z)$, the open set $C(A,Z) \cap S(y,V) = \{ f \in C(Y,Z) : f(\{y\} \cap A) \in V \} = \{ f \in C(Y,Z) : f|_A(y) \in V \} = S(y,V)$ in $C_\zeta(A,Z)$ contains $f|_A(y)$ but not $g|_A(y)$. Therefore the space $C_\zeta(A,Z)$ is a $T_\circ$ space. □

Theorem 2.2 *Let the function space $C_\tau(Y,Z)$ be a T_1 space. The function space $C_\zeta(A,Z)$ for $A \subset Y$ is a T_1 space.*

Proof. Let $f, g \in C_\tau(Y,Z)$ be distinct maps such that $\forall y \in Y$, $f(y) \neq g(y)$. Then $\forall y \in A$,

$f|_A(y) \neq g|_A(y)$. For the open sets $S(y,V)$ containing f but not g and $S(y,U)$ containing g but not f in $C_\tau(Y,Z)$, the open sets

$$C(A,Z) \cap S(y,V) = \{f \in C(Y,Z): f(\{y\} \cap A) \in V\} = \{f \in C(Y,Z): f|_A(y) \in V\} = S(y,V)$$

and

$$C(A,Z) \cap S(y,U) = \{g \in C(Y,Z): g(\{y\} \cap A) \in U\} = \{g \in C(Y,Z): g|_A(y) \in U\} = S(y,U)$$

in $C_\zeta(A,Z)$ are neighborhoods of $f|_A(y)$ but not $g|_A(y)$ and $g|_A(y)$ but not $f|_A(y)$ respectively. Therefore the space $C_\zeta(A,Z)$ is a T_1 space. □

Theorem 2.3 *Let the function space* $C_\tau(Y,Z)$ *be a* T_2 *space. The function space* $C_\zeta(A,Z)$ *for* $A \subset Y$ *is a* T_2 *space.*

Proof. Let $f, g \in C_\tau(Y,Z)$ be distinct maps such that $\forall y \in Y$, $f(y) \neq g(y)$. Then $\forall y \in A$, $f|_A(y) \neq g|_A(y)$. For the disjoint open sets $S(y,V)$ and $S(y,U)$ neighborhoods of f and g respectively in $C_\tau(Y,Z)$, the open sets

$$C(A,Z) \cap S(y,V) = \{f \in C(Y,Z): f(\{y\} \cap A) \in V\} = \{f \in C(Y,Z): f|_A(y) \in V\} = S(y,V)$$

and

$$C(A,Z) \cap S(y,U) = \{g \in C(Y,Z): g(\{y\} \cap A) \in U\} = \{g \in C(Y,Z): g|_A(y) \in U\} = S(y,U)$$

in $C_\zeta(A,Z)$ are disjoint neighborhoods of $f|_A(y)$ and $g|_A(y)$ respectively. Therefore the space $C_\zeta(A,Z)$ is a T_2 space. □

Theorem 2.4 *Let the function space* $C_\tau(Y,Z)$ *be a regular space for a regular space* Z. *The function space* $C_\zeta(A,Z)$ *for* $A \subset Y$ *is a regular space.*

Proof. The space $C_\tau(Y,Z)$ is regular for a regular space Z if for the open cover $F(K_i,U_i)$ of f, there exist open sets $F(K_i,V_i) \subset F(K_i,U_i)$ neighborhoods of f such that for $g \in C(Y,Z)$ and $g \notin F(K_i,U_i)$, $F(x, Z \setminus V_i)$ for some $x \in K_i$ is a neighborhood of g which does not intersect $F(K_i,V_i)$ and $\overline{F(K_i,V_i)} \subset F(K_i,U_i)$. For $F(K_i,V_i) \subset F(K_i,U_i)$, $F(K_i,V_i) \cap C(A,Z) \subset F(K_i,U_i) \cap C(A,Z)$ implying that $F(\mathsf{U}_i,V_i) \subset F(\mathsf{U}_i,U_i)$, where $\mathsf{U}_i = K_i \cap A$. For $g \notin F(K_i,U_i)$ we have that $g \notin F(\mathsf{U}_i,U_i)$, implying that $g|_A \notin F(\mathsf{U}_i,U_i)$ and for $x \in A$, $g|_A \in F(x, Z \setminus V_i)$. Therefore $F(x, Z \setminus V_i)$ is a neighbourhood of $g|_A$ not intersecting $F(\mathsf{U}_i,V_i)$. $F(K_i,V_i) \cap C(A,Z) \subset F(K_i,\overline{V_i}) \cap C(A,Z) \subset F(K_i,U_i) \cap C(A,Z)$ implies that $F(\mathsf{U}_i,V_i) \subset F(\mathsf{U}_i,\overline{V_i}) \subset F(\mathsf{U}_i,U_i)$. From the assertion $\overline{F(K_i,V_i)} \subset F(K_i,\overline{V_i})$ in Lemma 1.4, we have that $\overline{F(\mathsf{U}_i,V_i)} \subset F(\mathsf{U}_i,\overline{V_i})$. Therefore $F(x, Z \setminus V_i)$ and $F(\mathsf{U}_i,U_i)$ are two disjoint open sets neighborhoods of $g|_A$ and $\overline{F(\mathsf{U}_i,V_i)}$ respectively. Hence the set $C(A,Z)$ with the induced topology ζ is a regular space. □

3. Conclusion

The underlying function space $C_\zeta(A,Z)$ inherits the T_i-separation axioms for $i = 0,1,2,3$ from the function space $C_\tau(Y,Z)$. From theorem 1.5, the underlying function space $C_\zeta(A,Z)$ is homeomorphic to the subspace $C_\varrho(\mathsf{U}_\circ, \mathsf{V}_\circ)$ of the function space $C_\tau(Y,Z)$. This implies that the subspace $C_\varrho(\mathsf{U}_\circ, \mathsf{V}_\circ)$ is a T_i-space for $i = 0,1,2,3$, if the function space $C_\tau(Y,Z)$ is a T_i-space for $i = 0,1,2,3$. Therefore the T_i-separation axioms for $i = 0,1,2,3$ are hereditary on function spaces.

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