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UNCONDITIONAL BANACH SPACE IDEAL PROPERTY

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Abstract

Let $L^{w'}$ denote the assignment which associates with each pair of Banach spaces X, Y, the vector space $L^{w'}(X, Y)$ and $K(X, Y)$ be the space of all compact linear operators from X to Y. Let $T \in L^{w'}(X, Y)$ and suppose $(T_n) \subset K(X, Y)$ converges in the dual weak operator topology (w') of T. Denote by $K_u((T_n))$ the finite number given by

$$K_u((T_n)) := \sup_{n \in \mathbf{N}} \{ \max \{\|T_n\|, \|T - 2T_n\|\}\}.$$

The u-norm on $L^{w'}(X, Y)$ is then given by

$$\|T\|_u := \inf\{K_u((T_n)) : T = w' - \lim_n T_n, \quad T_n \in K(X, Y)\}.$$

It has been shown that $(L^{w'}(X, Y)\|.\|_u)$ is a Banach operator ideal. We find conditions for $K(X, Y)$ to be an unconditional ideal in $(L^{w'}(X, Y)\|.\|_u)$.

1. Introduction

In Section 8 of paper [2], the authors established necessary conditions on a Banach space X such that the space $K(X)$ of compact operators is a u-ideal in the space $L(X)$ of bounded linear operators, showing that this is the case if X is separable and has (UKAP) (unconditional compact approximation property, i.e., if there exists a sequence (K_n) in $K(X)$ such that $\lim_n K_n x = x$ for all $x \in X$ and $\lim_n \|id_X - 2K_n\| = 1$).

Johnson proved in [5] that if Y is a Banach space having the bounded approximation property, then the annihilator $K(X, Y)^{\perp}$ in the (continuous) dual space $L(X, Y)^*$ is the kernel of a projection on $L(X, Y)^*$. The range space of the projection is isomorphic to the dual space $K(X, Y)^*$. John showed in [3] that Johnson's result is also true in case of any separable Pisier space $X = P$ and its dual $Y = P^*$, both being spaces, which do not have the approximation property. This motivated his more general results in a later paper (cf. [4]).

In the paper [1], an alternative (operator ideal) approach is followed to prove similar (and more general) versions of John's results. Having proved that $(L^{w'}, \|.\|_u)$ is a Banach operator ideal (cf. [6]), we shall build on the results in [1] to obtain conditions for the space $K(X, Y)$ of compact operators to be a u-ideal in a suitable subspace $(L^{w'}(X, Y), \|.\|_u)$ of $L(X, Y)$. If $(L^{w'}(X, Y)) = L(X, Y)$, our results states conditions on $L(X, Y)$ so that $K(X, Y)$ is a u-ideal in $L(X, Y)$.

Before we investigate the u-ideal property of $K(X, Y)$ in $(L^{w'}(X, Y), \|.\|_u)$, we recall from [1], the ideal property of $K(X, Y)$ in $(L^{w'}(X, Y))$ with respect to the $|\|.\||$-norm.

Theorem 1.1 (cf. [1], Theorem 2.5)**.** *There exists a continuous bilinear form*

$$J : L^{w'}(X, Y)^* \times L^{w'}(X, Y) \to \boldsymbol{K},$$

such that

(a) $J(\phi, T) = \phi(T)$ *for all* $(\phi, T) \in L^{w'}(X, Y)^* \times L^{w'}(X, Y)$;

(b) $|J(\phi, T)| \leq \|\phi\| |\|T\||$ *for all* $T \in L^{w'}(X, Y)$ *and* $\phi \in L^{w'}(X, Y)^*$;

(c) $J(\phi, T) = \lim_n \phi(T_n)$, *where* (T_n) *is any sequence of compact operators* $T_n \in K(X, Y)$ *tending to T in* w'*-topology.*

Corollary 1.2. *Let* X, Y *be Banach spaces. There is a projection*

$$P : (L^{w'}(X, Y), |\|.\||)^* \to (L^{w'}(X, Y), |\|.\||)^*,$$

such that $\mathrm{Ker}(P) = K(X, Y)^{\perp} = \{\phi \in L^{w'}(X, Y)^{*} : \phi \setminus K(X, Y) = 0\}$, $\|P\| \leq 1$ *and the range of P is isomorphic to* $K(X, Y)^{*}$. *Thus* $K(X, Y)$ *is an ideal in* $(L^{w'}(X, Y), |\|.\||)$. *The projection P is given by*

$$P\phi(T) = \lim_{n} \phi(T_n) = J(\phi, T),$$

for all $\phi \in L^{w'}(X, Y)^{*}$ *and* $T \in L^{w'}(X, Y)$.

Since the norms $\|.\|$ and $|\|.\||$ are equivalent when $L(X, Y) = L^{w'}(X, Y)$, it follows from ([6], Corollary 2.7) that

Corollary 1.3 (cf. [4]). *Let* X, Y *be Banach spaces such that for each* $T \in L(X, Y)$, *there is a sequence* $(T_n) \subset K(X, Y)$ *such that* $\underset{\rightarrow}{w'} T$. *Then there exists a projection*

$$P : L(X, Y)^{*} \rightarrow L(X, Y)^{*},$$

such that

$$\mathrm{Ker}(P) = K(X, Y)^{\perp} = \{\phi \in L^{w'}(X, Y)^{*} : \phi \setminus K(X, Y) = 0\},$$

and the range of P is isomorphic to $K(X, Y)^{*}$.

2. Unconditional Ideal Property

The authors in [2] call a sequence (K_n) of compact operators from X into X *a compact approximation sequence*, if $\lim_{n} K_n x = x$ for all $x \in X$. In [2], it is also agreed to say that X has (UKAP), if there is a compact approximation sequence $K_n : X \rightarrow X$ such that $\lim_{n\to\infty} \|I - 2K_n\| = 1$. It is also proved in [2] that if X is a separable Banach space, then X has (UKAP), if and only if for every $\epsilon > 0$, there is a sequence (A_n) of compact operators such that for every $x \in X$ and every n and every $\theta_j = \pm 1, 1 \le j \le n,$ we have $\sum_{i=1}^{\infty} A_n x = x$ and $\|\sum_{i=1}^{\infty} \theta_j A_j x\| \le (1 + \epsilon)\|x\|.$ In particular, this means that if we let $K_n = \sum_{i=1}^{\infty} A_i,$ then $K_n x \to x,$ $\forall x \in X$ and

$$\|K_n x\| \le (1 + \epsilon)\|x\|, \quad \forall x \in X, \quad \forall n \in \mathbf{N}.$$

Moreover, also $\|I - 2K_n\| \le 1 + \epsilon.$

When a separable Banach space X has UKAP, it is easily seen that for each $T \in L(X)$, $TK_n \to T(\text{as } n \to \infty)$ in the weak operator topology. If X is also reflexive, then $TK_n \to T(\text{as } n \to \infty)$ in the w'-topology and it follows that

$$K_u((TK_n)) \le (1 + \epsilon)\|T\|.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\|T\|_u \le \|T\|,$ i.e., $\|T\| = \|T\|_u$ in this case. Putting $T_n := TK_n,$ it follows that $T_n \overset{w'}{\to} T$ and

$$\|T - 2T_n\| \le (1 + \epsilon)\|T\|,$$

for all $n \in \mathbf{N}.$ Consequently, it follows that

$$\begin{aligned} \|Id_{(L^{w'})^*} - 2P\| &= \sup_{\|\phi\| \le 1} \|\phi - 2P\phi\| \\ &= \sup_{\|\phi\| \le 1} \sup_{\|T\| \le 1} |\phi(T) - 2P\phi(T)| \\ &= \sup_{\|\phi\| \le 1} \sup_{\|T\| \le 1} |\lim_n \phi(T - 2T_n)| \\ &\le \sup_{\|T\| \le 1} \sup_n \|T - 2T_n\| \le 1 + \epsilon. \end{aligned}$$

This being so for all $\epsilon > 0,$ it is clear that

Proposition 2.1 (Special case of [2], Proposition 8.2)**.** *Let X be a separable reflexive Banach space. If X has (UKAP), then $K(X)$ is a u-ideal in $L(X)$.*

If X satisfies the conditions in Proposition 2.1 and Y is any Banach space, then for each $T \in L(X, Y)$ and each $\epsilon > 0$, we may choose the sequence $(K_n) \subset K(X)$ to satisfy the properties in the above proof of Proposition 2.1. Again, put $T_n = TK_n$ for all n. Then, as before, $T_n \underset{\to}{w'} T$ and we still have the inequalities

$$\|T - 2T_n\| \leq (1 + \epsilon)\|T\| \text{ and } K_u((T_n)) \leq (1 + \epsilon)\|T\|.$$

Hence $\|T\|_u \leq (1 + \epsilon)\|T\|$ for all $\epsilon > 0$. The existence of a contractive projection $P : L(X, Y)^* \to L(X, Y)^*$ with $\text{Ker}(P) = K(X, Y)^{\perp}$ follows from the Theorem 1.1 and Corollary 1.2, since in this case, we have $(L(X, Y), \|.\|) = (L^{w'}(X, Y), \|.\|_u)$. Therefore, $K(X, Y)$ is an ideal in $L(X, Y)$. The argument in the proof of Proposition 2.1, then shows that

Proposition 2.2. *Let X be a separable reflexive Banach space and Y be any Banach space. If X has (UKAP), then $K(X, Y)$ is a u-ideal in $L(X, Y)$.*

In the above discussion of the proof of Proposition 2.1, it is important to realize that for each $T \in L(X, Y)$ and each $\epsilon > 0$, the sequence $(T_n) \subset K(X, Y)$ can be chosen to satisfy $T_n \underset{\to}{w'} T$ and $\|T - 2T_n\| \leq (1 + \epsilon)\|T\|$ and $\|T_n\| \leq (1 + \epsilon)\|T\|$. With the conditions on the Banach space X in Proposition 2.2, the norms $\|.\|$, $|||.|||$, and $\|.\|_u$ coincide on $L(X, Y)$, exactly because we can choose the sequence (T_n) as such. Therefore, it is natural to formulate the following definition:

Definition 2.3. Let X and Y be Banach spaces. We say an operator $T \in L(X, Y)$ has $(w' - \text{UKAP})$ (i.e., it has the "w'-uniform compact approximation property" if each $\epsilon > 0$, there exists a sequence $(T_n) \subset K(X, Y)$ such that $T = w' - \lim_n T_n$, $\|T - 2T_n\| \le (1 + \epsilon)\|T\|$, and $\|T_n\| \le (1 + \epsilon)\|T\|$ for all n. It follows from the above discussion that

Proposition 2.4. *Suppose each* $T \in (L^{w'}(X, Y), \|.\|_u)$ (*respectively, each* $T \in L(X, Y)$) *has* $(w' - UKAP)$. *Then* $K(X, Y)$ *is a u-ideal in* $(L^{w'}(X, Y), \|.\|_u)$ (*respectively, in* $L(X, Y)$).

Although Proposition 2.1 is here discussed as a motivation for the condition $(w' - \text{UKAP})$ in Proposition 2.4, it was already proved in [2] (cf. Theorem 8.3) that a separable reflexive Banach space has (UKAP), if and only if $K(X)$ is a u-ideal in $L(X)$. In this context, we may introduce yet another property on Banach spaces, as follows: A sequence (K_n) of compact operators from X into X is called a *w'-compact approximating sequence*, if $w' - \lim_n K_n = I$. If X is reflexive, then clearly each compact approximating sequence is w'-compact approximating. We say X has $(w' - \text{UKAP})$ if for each $\epsilon > 0$, there is a w'-compact approximating sequence $K_n : X \to X$ such that $\|K_n x\| \le (1 + \epsilon)\|x\|$, $\forall x \in X$, $\forall n \in \mathbf{N}$, and $\|I - 2K_n\| \le 1 + \epsilon$ for all n. It then follows from Proposition 2.4 that

Corollary 2.5. *If* X *has* $(w' - UKAP)$, *then* $K(X)$ *is a u-ideal in* $L(X)$.

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