**Pure and Applied Mathematics Journal**
2026; 15(2): 29-34
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20261502.13

# On $(\alpha, \beta)$–Almost Similar Operators in Hilbert Spaces

**Beatrice Obiero Adhiambo[1], Victor Wanjala[2, *]**

[1]Department of Mathematics and Computer, Rongo University, Rongo, Kenya
[2]Department of Mathematics and Physical Sciences, Maasai Mara University, Narok, Kenya

## Email address:
wanjalavictor421@gmail.com (Victor Wanjala)
[*]Corresponding author

## To cite this article:
Beatrice Obiero Adhiambo, Victor Wanjala. (2026). On $(\alpha, \beta)$–Almost Similar Operators in Hilbert Spaces. *Pure and Applied Mathematics Journal, 15*(2), 29-34. https://doi.org/10.11648/j.pamj.20261502.13

**Received:** 16 March 2026 ; **Accepted:** 25 March 2026 ; **Published:** 24 April 2026

**Abstract:** This paper introduces and investigates a novel generalization of operator similarity, termed $(\alpha, \beta)$–almost similarity, which extends the concept of almost similar operators by incorporating two real parameters. We establish fundamental properties of this new equivalence relation, demonstrating that it forms an equivalence class on the space of bounded linear operators on a Hilbert space. Key results include the invariance of spectrum, point spectrum, and approximate point spectrum under this relation. The study also defines the class of $(\alpha, \beta)$-$\mathfrak{T}$ operators, an expansion of the classical $\mathfrak{T}$-operator concept, and explores its relationship with $(\alpha, \beta)$–almost similarity. Furthermore, we analyze the connections between similarity, unitary equivalence, and $(\alpha, \beta)$–almost similarity, providing conditions under which these relations coincide, particularly for self-adjoint and projection operators. The results contribute to the broader understanding of operator equivalence relations and their spectral implications.

**Keywords:** $\alpha$–almost Similar, Almost Similar, Unitarily Equivalent, Self-adjoint Operator

## 1. Introduction

We denote $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ the set of all bounded linear operators from a Hilbert space $\mathfrak{H}_1$ into a Hilbert space $\mathfrak{H}_2$. If $\mathfrak{H}_1 = \mathfrak{H}_2$, then we denote $\mathfrak{B}(\mathfrak{H})$ instead of $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. The operator $\mathfrak{A} \in \mathfrak{B}(\mathfrak{H})$ is called self-adjoint if $\mathfrak{A} = \mathfrak{A}^*$ where $\mathfrak{A}^*$ is the adjoint of $\mathfrak{A}$ [1].

An operator $\mathfrak{A} \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is said to be isometric if $\mathfrak{A}^*\mathfrak{A} = \mathfrak{I}_{\mathfrak{H}_1}$. If $\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A}$, then $\mathfrak{A}$ is called a normal operator. And if $\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A} = \mathfrak{I}$, then $\mathfrak{A}$ is said to be unitary [3]. If $\mathfrak{A}^2 = \mathfrak{A}$ and $\mathfrak{A} = \mathfrak{A}^*$, then $\mathfrak{A}$ is said to be a projection. If $\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}$ then $\mathfrak{A}$ is said to be partially isometric, equivalently $\mathfrak{A}^*\mathfrak{A}$ is a projection (that is, $(\mathfrak{A}^*\mathfrak{A})^2 = \mathfrak{A}^*\mathfrak{A}$) [4]. Clearly, every unitary operator is isometric and normal.

Two operators $\mathfrak{A}, \mathfrak{B} \in \mathfrak{B}(\mathfrak{H})$ are said to be similar and are denoted by $\mathfrak{A} \sim \mathfrak{B}$ if there exists an invertible operator $\mathfrak{X}$ such that $\mathfrak{X}\mathfrak{A}\mathfrak{X}^{-1} = \mathfrak{B}$ (equivalently, $\mathfrak{A} = \mathfrak{X}^{-1}\mathfrak{B}\mathfrak{X}$). If $\mathfrak{A} \sim \mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ have the same: spectrum, point spectrum, and approximate point spectrum [5].

Similarly, two operators $\mathfrak{A}, \mathfrak{B} \in \mathfrak{B}(\mathfrak{H})$ are said to be unitarily equivalent and are denoted by $\mathfrak{A} \cong \mathfrak{B}$, if there exists a unitary operator $\mathfrak{U}$ such that $\mathfrak{U}\mathfrak{A}\mathfrak{U}^{-1} = \mathfrak{B}$ (equivalently, $\mathfrak{A} = \mathfrak{U}^{-1}\mathfrak{B}\mathfrak{U}$). If $\mathfrak{A}, \mathfrak{B}$ are similar normal operators, then they are unitarily equivalent by the Fuglede–Putnam theorem [6].

Let $\mathfrak{A}, \mathfrak{B}$ be two bounded linear operators on $\mathfrak{B}(\mathfrak{H})$. Then $\mathfrak{A}, \mathfrak{B}$ are said to be $(\alpha, \beta)$–almost similar and denoted by $\mathfrak{A} \sim_{(\alpha,\beta)} \mathfrak{B}$ if there exists an invertible operator $\mathfrak{X}$ such that:

$$\mathfrak{A}^*\mathfrak{A} = \mathfrak{X}^{-1}\mathfrak{B}^*\mathfrak{B}\mathfrak{X}$$
$$\beta\mathfrak{A}^* + \alpha\mathfrak{A} = \mathfrak{X}^{-1}(\beta\mathfrak{B}^* + \alpha\mathfrak{B})\mathfrak{X}$$

The class of almost similar operators was first introduced by Jibril [7]. We have extended this concept to $(\alpha, \beta)$–almost similar and demonstrated some different results.

An operator $\mathfrak{T} \in \mathfrak{B}(\mathfrak{H})$ is said to be a $\mathfrak{T}$-operator if $\mathfrak{T}$ commutes with $\mathfrak{T}^*\mathfrak{T}$. The class of all $\mathfrak{T}$-operators in $\mathfrak{B}(\mathfrak{H})$ is denoted by $\mathcal{T}$. The class of $\mathfrak{T}$-operators has been widely studied by Campbell [8]. We have extended the concept of $\mathfrak{T}$-operator to another concept we called it $(\alpha, \beta)$-$\mathfrak{T}$ operator, the class of $(\alpha, \beta)$-$\mathfrak{T}$ operators in $\mathfrak{B}(\mathfrak{H})$ is denoted by $\mathcal{T}_{(\alpha,\beta)}$.

Let $\mathfrak{T} \in \mathfrak{B}(\mathfrak{H})$ then the set of all complex numbers $\lambda$ for which $\lambda\mathfrak{I} - \mathfrak{T}$ is not invertible is called the spectrum of $\mathfrak{T}$ and denoted by $\sigma(\mathfrak{T})$, that is,

$$\sigma(\mathfrak{T}) = \{\lambda \in \mathbb{C} : (\lambda\mathfrak{I} - \mathfrak{T}) \text{ is not invertible}\}$$

The complement of the spectrum of $\mathfrak{T}$ is called the resolvent set of $\mathfrak{T}$. The spectrum of $\mathfrak{T}$ can be split into many disjoint sets [9].

The point spectrum of the operator $\mathfrak{T}$ is denoted by $\sigma_p(\mathfrak{T})$ and is the set of all those $\lambda$ for which $\lambda\mathfrak{I} - \mathfrak{T}$ is not injective, that is

$$\sigma_p(\mathfrak{T}) = \{\lambda \in \mathbb{C} : (\lambda\mathfrak{I} - \mathfrak{T})x = 0,\ x \neq 0\}$$

A scalar $\lambda$ is said to be in the approximate point spectrum of the operator $\mathfrak{T}$, denoted by $\sigma_{ap}(\mathfrak{T})$, if there exists a sequence of unit vectors $\{\mathfrak{x}_n\}$ such that $\|(\lambda\mathfrak{I} - \mathfrak{T})\mathfrak{x}_n\| \to 0$ [9].

Let $\mathfrak{T}$ be a linear transformation from a normed space $\mathfrak{X}$ into a normed space $\mathfrak{Y}$, $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X},\mathfrak{Y})$. Then $\mathfrak{T}$ is said to be compact if $\mathfrak{T}(\mathfrak{S})$ is compact for every bounded subset $\mathfrak{S}$ of $\mathfrak{X}$. That is, $\mathfrak{T}(\mathfrak{S})$ is relatively compact for every bounded subset $\mathfrak{S}$ of $\mathfrak{X}$ [9].

## 2. Basic Concept on $\alpha$–almost Similarity

*Definition* 2.1. Let $\alpha, \beta \in \mathbb{R}$. Two bounded linear operators $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}(\mathfrak{H})$ are said to be $(\alpha,\beta)$–almost similar and are denoted by $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$ if there exists an invertible operator $\mathfrak{X}$ such that:

$$\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{X}^{-1}\mathfrak{T}_2^*\mathfrak{T}_2\mathfrak{X} \tag{1}$$
$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = \mathfrak{X}^{-1}(\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2)\mathfrak{X} \tag{2}$$

*Example* 2.1. Let

$$\mathfrak{T}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathfrak{T}_2 = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$

be operators on the two-dimensional Hilbert space $\mathbb{C}^2$, and define the invertible operator on $\mathbb{C}^2$ as follows:

$$\mathfrak{X} = \mathfrak{X}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Take $\alpha = 2$, $\beta = 1$. Then $\mathfrak{T}_1 \sim_{(2,1)} \mathfrak{T}_2$. To show this, we compute:

$$\mathfrak{T}_1^*\mathfrak{T}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathfrak{X}^{-1}\mathfrak{T}_2^*\mathfrak{T}_2\mathfrak{X} = \begin{bmatrix} 9 & 6 \\ 6 & 20 \end{bmatrix}$$

$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}, \quad \mathfrak{X}^{-1}(\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2)\mathfrak{X} = \begin{bmatrix} 5 & 2 \\ 0 & 6 \end{bmatrix}$$

*Remark* 2.1. Every $(1,1)$–almost similar pair of operators is almost similar, and the converse is also true.

The following example shows that almost similarity and $(\alpha,\beta)$–almost similarity are independent when $(\alpha,\beta) \neq (1,1)$.

*Example* 2.2. Let

$$\mathfrak{T}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathfrak{T}_2 = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$

be operators on the two-dimensional Hilbert space $\mathbb{C}^2$, and define the invertible operator on $\mathbb{C}^2$ as:

$$\mathfrak{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Take $\alpha = 2$, $\beta = 1$. Then $\mathfrak{T}_1 \sim_{(2,1)} \mathfrak{T}_2$. However, $\mathfrak{T}_1$ is not almost similar to $\mathfrak{T}_2$, since

$$\mathfrak{T}_1^* + \mathfrak{T}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \neq \mathfrak{X}^{-1}(\mathfrak{T}_2^* + \mathfrak{T}_2)\mathfrak{X} = \begin{bmatrix} 6 & 2 \\ 2 & 8 \end{bmatrix}$$

*Theorem* 2.1. Let $\alpha, \beta \in \mathbb{R}$. The relation $\sim_{(\alpha,\beta)}$ on $\mathfrak{B}(\mathfrak{H})$ is an equivalence relation.

*Proof:*

1. *Reflexivity*: Let $\mathfrak{T}_1 \in \mathfrak{B}(H)$ and take $\mathfrak{X} = \mathfrak{I}$. Then clearly,
$$\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{X}^{-1}\mathfrak{T}_1^*\mathfrak{T}_1\mathfrak{X}$$
$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = \mathfrak{X}^{-1}(\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1)\mathfrak{X}$$
Hence, $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_1$.
2. *Symmetry*: Suppose $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}(H)$ and $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$. Then there exists an invertible operator $\mathfrak{X}$ such that:
$$\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{X}^{-1}\mathfrak{T}_2^*\mathfrak{T}_2\mathfrak{X} \tag{3}$$
$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = \mathfrak{X}^{-1}(\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2)\mathfrak{X} \tag{4}$$
Premultiplying and postmultiplying both sides of (3) and (4) by $\mathfrak{X}$ and $\mathfrak{X}^{-1}$, respectively, yields:
$$\mathfrak{T}_2^*\mathfrak{T}_2 = \mathfrak{X}\mathfrak{T}_1^*\mathfrak{T}_1\mathfrak{X}^{-1} \tag{5}$$
$$\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2 = \mathfrak{X}(\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1)\mathfrak{X}^{-1} \tag{6}$$
Let $\mathfrak{Y} = \mathfrak{X}^{-1}$, which is also invertible. Substituting gives $\mathfrak{T}_2 \sim_{(\alpha,\beta)} \mathfrak{T}_1$.
3. *Transitivity*: Suppose $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3 \in \mathfrak{B}(H)$ with $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$ and $\mathfrak{T}_2 \sim_{(\alpha,\beta)} \mathfrak{T}_3$.
Since $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$, there exists an invertible operator $\mathfrak{X}_1$ such that:
$$\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{X}_1^{-1}\mathfrak{T}_2^*\mathfrak{T}_2\mathfrak{X}_1 \tag{7}$$
$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = \mathfrak{X}_1^{-1}(\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2)\mathfrak{X}_1 \tag{8}$$
Similarly, since $\mathfrak{T}_2 \sim_{(\alpha,\beta)} \mathfrak{T}_3$, there exists an invertible operator $\mathfrak{X}_2$ such that:
$$\mathfrak{T}_2^*\mathfrak{T}_2 = \mathfrak{X}_2^{-1}\mathfrak{T}_3^*\mathfrak{T}_3\mathfrak{X}_2 \tag{9}$$
$$\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2 = \mathfrak{X}_2^{-1}(\beta\mathfrak{T}_3^* + \alpha\mathfrak{T}_3)\mathfrak{X}_2 \tag{10}$$
Substituting (9) and (10) into (7) and (8) gives:
$$\mathfrak{T}_1^*\mathfrak{T}_1 = (\mathfrak{X}_2\mathfrak{X}_1)^{-1}\mathfrak{T}_3^*\mathfrak{T}_3(\mathfrak{X}_2\mathfrak{X}_1)$$

$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = (\mathfrak{X}_2\mathfrak{X}_1)^{-1}(\beta\mathfrak{T}_3^* + \alpha\mathfrak{T}_3)(\mathfrak{X}_2\mathfrak{X}_1)$$

Hence, $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_3$.

*Proposition* 2.1. Let $\mathfrak{T} \in \mathfrak{B}(H)$. If $\mathfrak{T} \sim_{(\alpha,\beta)} 0$, then $\mathfrak{T} = 0$.

*Proof:* Since $\mathfrak{T} \sim_{(\alpha,\beta)} 0$, there exists an invertible operator $\mathfrak{X}$ such that:

$$\mathfrak{T}^*\mathfrak{T} = \mathfrak{X}^{-1}(0^*0)\mathfrak{X} = 0 \tag{11}$$

$$\beta\mathfrak{T}^* + \alpha\mathfrak{T} = \mathfrak{X}^{-1}(\beta\cdot 0^* + \alpha\cdot 0)\mathfrak{X} = 0 \tag{12}$$

Thus $\mathfrak{T}^*\mathfrak{T} = 0$, which implies $\|\mathfrak{T}x\|^2 = \langle\mathfrak{T}x, \mathfrak{T}x\rangle = 0$ for all $x \in H$. Hence, $\mathfrak{T} = 0$.

*Remark* 2.2. Suppose that $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}(H)$ such that $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$. Then, by induction, one can prove:

1. $\sigma(\mathfrak{T}_1) = \sigma(\mathfrak{T}_2)$,
2. $\sigma_p(\mathfrak{T}_1) = \sigma_p(\mathfrak{T}_2)$ for all natural numbers $n$.

*Proposition* 2.2. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}(H)$ such that $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$. Then $\mathfrak{T}_1$ is isometric if and only if $\mathfrak{T}_2$ is isometric.

*Proof:* Suppose that $\mathfrak{T}_1$ is isometric. Since $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$, there exists an invertible operator $\mathfrak{X}$ such that:

$$\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{X}^{-1}\mathfrak{T}_2^*\mathfrak{T}_2\mathfrak{X} \tag{13}$$

$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = \mathfrak{X}^{-1}(\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2)\mathfrak{X} \tag{14}$$

Since $\mathfrak{T}_1$ is isometric, $\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{I}$. Substituting, we obtain $\mathfrak{T}_2^*\mathfrak{T}_2 = \mathfrak{I}$, i.e. $\mathfrak{T}_2$ is isometric.

Conversely, the same reasoning shows that $\mathfrak{T}_1$ is isometric whenever $\mathfrak{T}_2$ is isometric.

*Proposition* 2.3. Let $\alpha, \beta \in \mathbb{R}$ and $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}(H)$ with $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$. Then:

1. $\mathfrak{T}_1$ is onto if and only if $\mathfrak{T}_2$ is onto,
2. $\mathfrak{T}_1^*$ is onto if and only if $\mathfrak{T}_2^*$ is onto,
3. $\mathfrak{T}_1$ is one-to-one if and only if $\mathfrak{T}_2$ is one-to-one,
4. $\mathfrak{T}_1^*$ is one-to-one if and only if $\mathfrak{T}_2^*$ is one-to-one,
5. $\mathfrak{T}_1$ is a projection if and only if $\mathfrak{T}_2$ is a projection.

*Proof:* Clearly.

*Remark* 2.3. Let $\alpha, \beta \in \mathbb{R}$ and $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}(H)$ with $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$. Then:

6. $\mathfrak{T}_1$ is bijective if and only if $\mathfrak{T}_2$ is bijective,
7. $\mathfrak{T}_1^*$ is bijective if and only if $\mathfrak{T}_2^*$ is bijective.

*Proof:* Immediate from the preceding proposition.

*Proposition* 2.4. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}(H)$ with $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$. Then $\mathfrak{T}_1^*\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2^*\mathfrak{T}_2$.

*Proof:* Since $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$, there exists an invertible operator $\mathfrak{X}$ such that

$$\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{X}^{-1}\mathfrak{T}_2^*\mathfrak{T}_2\mathfrak{X}$$

$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = \mathfrak{X}^{-1}(\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2)\mathfrak{X}$$

Hence, $(\mathfrak{T}_1^*\mathfrak{T}_1) \sim_{(\alpha,\beta)} (\mathfrak{T}_2^*\mathfrak{T}_2)$.

*Proposition* 2.5. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{B}(H)$ with $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$. If $\mathfrak{T}_1$ is partially isometric, then so is $\mathfrak{T}_2$.

*Proof:* Since $\mathfrak{T}_1 \sim_{(\alpha,\beta)} \mathfrak{T}_2$, there exists an invertible operator $\mathfrak{X}$ such that

$$\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{X}^{-1}\mathfrak{T}_2^*\mathfrak{T}_2\mathfrak{X}$$

$$\beta\mathfrak{T}_1^* + \alpha\mathfrak{T}_1 = \mathfrak{X}^{-1}(\beta\mathfrak{T}_2^* + \alpha\mathfrak{T}_2)\mathfrak{X}$$

If $\mathfrak{T}_1$ is partially isometric, then $\mathfrak{T}_1\mathfrak{T}_1^*\mathfrak{T}_1 = \mathfrak{T}_1$. Substituting into the above relations, it follows that $\mathfrak{T}_2\mathfrak{T}_2^*\mathfrak{T}_2 = \mathfrak{T}_2$, i.e. $\mathfrak{T}_2$ is also partially isometric.

*Proposition* 2.6. Let $\alpha, \beta \in \mathbb{R}$. Then the transformation $\phi : \mathfrak{B}(\mathfrak{H}) \to \mathfrak{B}(\mathfrak{H})$ that satisfies $\phi(\mathfrak{T}) = \mathfrak{X}\mathfrak{T}\mathfrak{X}^{-1}$, $\phi(\mathfrak{T}^*) = \mathfrak{X}\mathfrak{T}^*\mathfrak{X}^{-1}$ is an automorphism. That is, it maps sums into sums, products into products and scalar multiples into scalar multiples.

*Proof:* Suppose that $\mathfrak{T}, \mathfrak{S} \in \mathfrak{B}(\mathfrak{H})$ and $\phi(\mathfrak{T}) = \mathfrak{X}\mathfrak{T}\mathfrak{X}^{-1}$ and $\phi(\mathfrak{S}) = \mathfrak{X}\mathfrak{S}\mathfrak{X}^{-1}$. Then:

$$\phi(\mathfrak{T} + \alpha\mathfrak{S}) = \mathfrak{X}(\mathfrak{T} + \alpha\mathfrak{S})\mathfrak{X}^{-1} = \mathfrak{X}\mathfrak{T}\mathfrak{X}^{-1} + \alpha\mathfrak{X}\mathfrak{S}\mathfrak{X}^{-1} = \phi(\mathfrak{T}) + \alpha\phi(\mathfrak{S})$$

and

$$\phi(\mathfrak{T}\mathfrak{S}) = \mathfrak{X}(\mathfrak{T}\mathfrak{S})\mathfrak{X}^{-1} = \mathfrak{X}\mathfrak{T}\mathfrak{X}^{-1}\mathfrak{X}\mathfrak{S}\mathfrak{X}^{-1} = \phi(\mathfrak{T})\phi(\mathfrak{S})$$

*Proposition* 2.7. Let $\mathfrak{T}, \mathfrak{S} \in \mathfrak{B}(\mathfrak{H})$ such that $\mathfrak{T}, \mathfrak{S}$ are unitarily equivalent. Then $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{S}$ for every $\alpha, \beta \in \mathbb{R}$.

*Proof:* Since $\mathfrak{T}$ and $\mathfrak{S}$ are unitarily equivalent, there exists a unitary operator $\mathfrak{U}$ such that $\mathfrak{U}\mathfrak{T}\mathfrak{U}^{-1} = \mathfrak{S}$. Then $\mathfrak{U}\mathfrak{U}^* = \mathfrak{I}$ which implies that $\beta\mathfrak{T}^* + \alpha\mathfrak{T} = \mathfrak{U}^{-1}(\beta\mathfrak{S}^* + \alpha\mathfrak{S})\mathfrak{U}$. Thus, $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{S}$ for all $\alpha, \beta \in \mathbb{R}$.

*Proposition* 2.8. Let $\mathfrak{T}, \mathfrak{S} \in \mathfrak{B}(\mathfrak{H})$ such that $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{S}$ for every real $\alpha, \beta$. Then $(\lambda\mathfrak{I} - \mathfrak{T}) \sim_{(\alpha,\beta)} (\lambda\mathfrak{I} - \mathfrak{S})$ for every real $\lambda$.

*Proof:* $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{S}$ means that there exists an invertible operator $\mathfrak{X}$ such that:

$$\mathfrak{T}^*\mathfrak{T} = \mathfrak{X}^{-1}\mathfrak{S}^*\mathfrak{S}\mathfrak{X} \tag{1}$$

$$\beta\mathfrak{T}^* + \alpha\mathfrak{T} = \mathfrak{X}^{-1}(\beta\mathfrak{S}^* + \alpha\mathfrak{S})\mathfrak{X} \tag{2}$$

Then we have

$$\mathfrak{X}(\lambda\mathfrak{I} - \mathfrak{T})\mathfrak{X}^{-1} = \lambda\mathfrak{I} - \mathfrak{X}\mathfrak{T}\mathfrak{X}^{-1} = \lambda\mathfrak{I} - \mathfrak{S}$$

which implies that:

$$(\lambda\mathfrak{I} - \mathfrak{T}) \sim_{(\alpha,\beta)} (\lambda\mathfrak{I} - \mathfrak{S}) \tag{3}$$

Since $\lambda$ is real, we also note that

$$(\lambda\mathfrak{I} - \mathfrak{T})^*(\lambda\mathfrak{I} - \mathfrak{T}) = ((\lambda\mathfrak{I} - \mathfrak{T}^*)(\lambda\mathfrak{I} - \mathfrak{T}))^*. \tag{4}$$

From (3) and (4) we conclude $(\lambda\mathfrak{I} - \mathfrak{T}) \sim_{(\alpha,\beta)} (\lambda\mathfrak{I} - \mathfrak{S})$ for every real $\lambda$.

*Proposition* 2.9. Let $\mathfrak{T}, \mathfrak{S} \in \mathfrak{B}(\mathfrak{H})$ be projections such that $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{S}$ and $\sigma(\mathfrak{T}) = \sigma(\mathfrak{S})$. Then:

$$\dim \mathfrak{N}(\mathfrak{T}) = \dim \mathfrak{N}(\mathfrak{S})$$
$$\dim \mathfrak{R}(\mathfrak{T}) = \dim \mathfrak{R}(\mathfrak{S})$$
$$\dim \mathfrak{N}(\mathfrak{T}^*) = \dim \mathfrak{N}(\mathfrak{S}^*)$$

*Proof:* $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{S}$ means that there exists an invertible operator $\mathfrak{X}$ such that:

$$\mathfrak{T}^*\mathfrak{T} = \mathfrak{X}^{-1}\mathfrak{S}^*\mathfrak{S}\mathfrak{X} \quad (1)$$
$$\beta\mathfrak{T}^{+}\alpha\mathfrak{T} = \mathfrak{X}^{-1}(\beta\mathfrak{S}^* + \alpha\mathfrak{S})\mathfrak{X} \quad (2)$$

Since $\mathfrak{T}$ and $\mathfrak{S}$ are projections, they are self-adjoint. Hence $\mathfrak{X}$ must be unitary, which implies:

$$\dim \mathfrak{N}(\mathfrak{T}) = \dim \mathfrak{N}(\mathfrak{S})$$
$$\dim \mathfrak{R}(\mathfrak{T}) = \dim \mathfrak{R}(\mathfrak{S})$$
$$\dim \mathfrak{N}(\mathfrak{T}^*) = \dim \mathfrak{N}(\mathfrak{S}^*)$$

*Theorem* 2.2 (10). The operator $\mathfrak{T} \in \mathfrak{B}(\mathfrak{H})$ is compact if and only if $\mathfrak{T}^*$ is compact.

*Proposition* 2.10. Let $\alpha,\beta \in \mathbb{R}$, $\mathfrak{T}, \mathfrak{b} \in \mathfrak{B}(H)$ and $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{b}$. If $\mathfrak{T}$ is compact then $\mathfrak{b}$ is compact.

*Proof:* Since $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{b}$ then there exists an invertible operator $\mathfrak{X}$ such that

$$\mathfrak{T}^*\mathfrak{T} = \mathfrak{X}^{-1}\mathfrak{b}^*\mathfrak{b}\,\mathfrak{X}$$

Pre-multiplying and post-multiplying both sides by $\mathfrak{X}^{-1}$ and $\mathfrak{X}$ respectively, we have $\mathfrak{T} = \mathfrak{X}^{-1}\mathfrak{b}\,\mathfrak{X}$. Since $\mathfrak{b}$ is compact then $\mathfrak{X}^{-1}\mathfrak{b}\,\mathfrak{X}$ is also compact. By theorem 1.17 above then $\mathfrak{T}$ is compact.

*Theorem* 2.3. Let $\alpha,\beta \in \mathbb{R}$, $\mathfrak{T}, \mathfrak{b} \in \mathfrak{B}(H)$, $\mathfrak{X}$ be an invertible operator. If

$$\mathfrak{T}^*\mathfrak{T} = \mathfrak{X}^{-1}\mathfrak{b}^*\mathfrak{b}\,\mathfrak{X}$$

$$\beta\mathfrak{T}^* + \alpha\mathfrak{T} = \mathfrak{X}^{-1}(\beta\mathfrak{b}^* + \alpha\mathfrak{b})\mathfrak{X}$$

then $\mathfrak{T}$ and $\mathfrak{b}$ are $(\alpha,\beta)$–almost similar.

*Proof:* By hypothesis $\mathfrak{T}^*\mathfrak{T} = \mathfrak{X}^{-1}\mathfrak{b}^*\mathfrak{b}\,\mathfrak{X}$ and $\beta\mathfrak{T}^* + \alpha\mathfrak{T} = \mathfrak{X}^{-1}(\beta\mathfrak{b}^* + \alpha\mathfrak{b})\mathfrak{X}$. Therefore, $\mathfrak{T}$ and $\mathfrak{b}$ are $(\alpha,\beta)$–almost similar.

*Proposition* 2.11. If $\mathfrak{T}, \mathfrak{b} \in \mathfrak{B}(H)$ are similar normal operators, then $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{b}$.

*Proof:* Suppose that $\mathfrak{T}$ and $\mathfrak{b}$ are similar normal operators, then there exists an invertible operator $\mathfrak{X}$ such that

$$\mathfrak{T}^*\mathfrak{T} = \mathfrak{X}^{-1}\mathfrak{b}^*\mathfrak{b}\,\mathfrak{X}$$

Then by Fuglede–Putnam theorem [6],

$$\mathfrak{T} + \mathfrak{T}^* = \mathfrak{X}^{-1}(\mathfrak{b} + \mathfrak{b}^*)\mathfrak{X}$$

Now, by using theorem 1.20 we have that $\mathfrak{T}$ and $\mathfrak{b}$ are $(\alpha,\beta)$–almost similar.

*Remark* 2.4. The converse of Proposition 1.20 is not true in general. Consider the following example: Let

$$\mathfrak{T} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}, \quad \mathfrak{b} = \begin{bmatrix}3 & 2\\ 0 & 4\end{bmatrix}, \quad \mathfrak{X} = \mathfrak{X}^{-1} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

be operators on the two-dimensional Hilbert space $\mathbb{C}^2$. Take $\alpha = 2, \beta = 1$, then $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{b}$. Also $\mathfrak{T}$ is similar to $\mathfrak{b}$ (i.e. $\mathfrak{T} \sim \mathfrak{b}$) but

$$\mathfrak{T}\mathfrak{T}^* = \begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix} \neq \mathfrak{T}^*\mathfrak{T}, \qquad \mathfrak{b}\mathfrak{b}^* = \begin{bmatrix}13 & 6\\ 6 & 20\end{bmatrix} \neq \mathfrak{b}^*\mathfrak{b}$$

Thus, $\mathfrak{T}$ and $\mathfrak{b}$ are not normal operators.

*Proof:* If $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{T}^*$ then clearly $\sigma(\mathfrak{T}) = \sigma(\mathfrak{T}^*)$ and $\mathfrak{T}$ is normal.

Conversely: If $\sigma(\mathfrak{T}) = \sigma(\mathfrak{T}^*)$ and $\mathfrak{T}$ is normal, then $(\mathfrak{T}^*\mathfrak{T})^{1/2} = (\mathfrak{T}\mathfrak{T}^*)^{1/2}$. Hence,

$$(\mathfrak{T}^*\mathfrak{T} + \alpha I)^{1/2} = (\mathfrak{T}\mathfrak{T}^* + \alpha I)^{1/2}$$

which implies that

$$(\mathfrak{T}^*\mathfrak{T} + \alpha I)^{1/2} = ((\mathfrak{T}^*)^*\mathfrak{T}^* + \alpha I)^{1/2}$$

Let $\|\mathfrak{T}x\|^2 = \langle \mathfrak{T}x|\mathfrak{T}x\rangle = \langle \mathfrak{T}^*\mathfrak{T}x|x\rangle = \|\mathfrak{T}^*x\|^2$ for every $x \in H$. Then for every $x \in H$: $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{T}^*$.

*Remark* 2.5. If $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{T}^*$ then $\sigma(\mathfrak{T}) = \sigma(\mathfrak{T}^*)$ for every $\alpha \neq 0$.

*Proposition* 2.12. Suppose that $\mathfrak{T} \in \mathfrak{B}(H)$ and $\sigma(\mathfrak{T}) = \sigma(\mathfrak{T}^*)$ then:

1. If $\alpha = 1$ and $\mathfrak{T}$ is normal then $\mathfrak{T} \sim_{(1,\beta)} \mathfrak{T}^*$.
2. If $\alpha = -1$ then $\mathfrak{T} \sim_{(-1,\beta)} \mathfrak{T}^*$.
3. If $\alpha \neq 1, -1$ then $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{T}^*$.

*Proof:* (i) Directly as in theorem 2.6. And (ii) clearly. Now to prove (iii), let $\alpha \neq 1, -1$. $\sigma(\mathfrak{T}) = \sigma(\mathfrak{T}^*)$. By taking adjoint to both sides we have $\sigma(\mathfrak{T}^*) = \sigma(\mathfrak{T})$. Then $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{T}^*$.

*Theorem* 2.4 (4). If $\mathfrak{T}$ is a normal operator, then there exists a unitary operator $\mathfrak{U}$ such that $\mathfrak{T} = \mathfrak{U}|\mathfrak{T}|$.

*Definition* 2.2. Let $\mathfrak{T} \in \mathfrak{B}(H)$, then $\mathfrak{T}$ is called an $(\alpha,\beta)$-$\mathfrak{T}$ operator if $\mathfrak{T}$ commutes with $\mathfrak{T}^*\mathfrak{T} + \alpha I$. The class of all $(\alpha,\beta)$-$\mathfrak{T}$ operators in a Banach algebra on a Hilbert space $H$ is denoted by $\mathcal{T}_{\alpha,\beta}$, i.e.

$$\mathcal{T}_{\alpha,\beta} = \{\mathfrak{T} \in \mathfrak{B}(H) : [\mathfrak{T}, \mathfrak{T}^*\mathfrak{T} + \alpha I] = 0\}$$

*Example* 2.3. Let $\mathfrak{T} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ and take $\alpha = 3$. Then

$$[\mathfrak{T}, \mathfrak{T}^*\mathfrak{T} + 3I] = 0$$

i.e. $\mathfrak{T}(\mathfrak{T}^*\mathfrak{T} + 3I) = (\mathfrak{T}^*\mathfrak{T} + 3I)\mathfrak{T}$, which implies that $\mathfrak{T}$ is a $(3,\beta)$-$\mathfrak{T}$ operator.

*Proposition* 2.13. If $\mathfrak{T} \in \mathfrak{B}(H)$ is an $(\alpha,\beta)$-$\mathfrak{T}$ operator then $\mathfrak{T}$ is a $(\gamma,\beta)$-$\mathfrak{T}$ operator for every real number $\gamma = \alpha$.

*Proof:* Clearly.

*Proposition* 2.14. If $\mathfrak{T}, \mathfrak{b} \in \mathfrak{B}(H)$ and $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{b}$ such that $\mathfrak{T}$ is an $(\alpha,\beta)$-$\mathfrak{T}$ operator then $\mathfrak{b}$ is also an $(\alpha,\beta)$-$\mathfrak{T}$ operator.

*Proof:* $\mathfrak{T} \sim_{(\alpha,\beta)} \mathfrak{b}$ means that there exists an invertible operator $\mathfrak{X}$ such that:

$$\mathfrak{T}^*\mathfrak{T} = \mathfrak{X}^{-1}\mathfrak{b}^*\mathfrak{b}\,\mathfrak{X} \quad (1)$$

$$\beta\mathfrak{T}^* + \alpha\mathfrak{T} = \mathfrak{X}^{-1}(\beta\mathfrak{b}^* + \alpha\mathfrak{b})\mathfrak{X} \quad (2)$$

Then,

$$[\mathfrak{T}, \mathfrak{T}^*\mathfrak{T} + \alpha I] = 0 \quad \Rightarrow \quad [\mathfrak{X}^{-1}\mathfrak{b}\mathfrak{X}, \mathfrak{X}^{-1}(\mathfrak{b}^*\mathfrak{b} + \alpha I)\mathfrak{X}] = 0$$

Hence $[\mathfrak{b}, \mathfrak{b}^*\mathfrak{b} + \alpha I] = 0$, which shows that $\mathfrak{b}$ is an $(\alpha, \beta)$-$\mathfrak{T}$ operator.

# 3. The Relation Among Similarity, Unitary Equivalence, Quasi Similarity and $(\alpha, \beta)$–almost Similarity

*Proposition* 3.1. Let $\mathfrak{T}, \mathfrak{b} \in \mathfrak{B}(H)$ be orthogonal projections. Then $\mathfrak{T}$ and $\mathfrak{b}$ are $(\alpha, \beta)$–almost similar if and only if $\mathfrak{T}$ and $\mathfrak{b}$ are similar.

*Proof:* Suppose that $A \sim_\alpha B$ and $A, B$ are projections. Then, by Proposition 1.16, we get $A \sim B$.

Conversely, suppose that $A$ and $B$ are similar operators. Then there exists an invertible operator $X$ such that $XAX^{-1} = B$. Since $A$ and $B$ are orthogonal projections, we have $A^2 = A = A^*$. Hence, $XX^* = I$.

On the other hand, the second inequality follows from the fact that

$$(A^*A + \alpha I)^{1/2}(B^*B + \alpha I)^{-1/2} = I$$

Thus, $A \sim_\alpha B$.

*Proposition* 3.2. Let $\alpha \in \mathbb{R}$ and $A, B \in B(H)$ be self-adjoint. Then $A$ and $B$ are unitarily equivalent if and only if $A \sim_\alpha B$.

*Proof:* Suppose that $A$ and $B$ are unitarily equivalent. Then, by Proposition 1.14, we have $A \sim_\alpha B$.

Conversely, suppose that $A, B \in B(H)$ are self-adjoint with $A \sim_\alpha B$. Now, $A \sim_\alpha B$ means that there exists an invertible operator $X$ such that:

$$XAX^{-1} = B \quad (1)$$

$$XX^* = (A^*A + \alpha I)^{1/2}(B^*B + \alpha I)^{-1/2} \quad (2)$$

Since $A, B$ are self-adjoint and $A \sim_\alpha B$, they are similar operators ($A \sim B$). Then $A$ and $B$ are both similar and self-adjoint, hence normal. Thus $A$ and $B$ are unitarily equivalent.

*Corollary* 3.1. Let $\alpha \in \mathbb{R}$, $A, B \in B(H)$ be self-adjoint, and $A \sim_\alpha B$. Then $A$ and $B$ are unitarily equivalent.

*Proof:* Directly from Proposition 4.2 above.

*Proposition* 3.3. Let $A, B \in B(H)$ be self-adjoint operators. Then $A$ and $B$ are $\alpha$-almost similar if and only if $A$ and $B$ are almost similar.

*Proof:* Suppose that $A, B$ are $\alpha$-almost similar. Then there exists an invertible operator $X$ such that:

$$XAX^{-1} = B \quad (1)$$

$$XX^* = (A^*A + \alpha I)^{1/2}(B^*B + \alpha I)^{-1/2} \quad (2)$$

Since $A$ and $B$ are self-adjoint, $A^* = A$ and $B^* = B$. Then (2) becomes:

$$(A^2 + \alpha I)^{1/2}(B^2 + \alpha I)^{-1/2} = XX^*$$

Now, pre-multiplying both sides by $(A^2 + \alpha I)^{-1/2}$ (with $\alpha \neq 0$) implies that:

$$XX^* = (A^2)^{1/2}(B^2)^{-1/2} \quad (3)$$

From (1) and (3), we have that $A$ and $B$ are almost similar.

Conversely, suppose that $A, B$ are almost similar. Then (1) and (3) hold. Since $A$ and $B$ are self-adjoint, (3) becomes

$$XX^* = (A^*A)^{1/2}(B^*B)^{-1/2}$$

Pre-multiplying both sides by $(A^*A + \alpha I)^{1/2}(A^*A)^{-1/2}$ implies that

$$(A^*A+\alpha I)^{1/2}(A^*A)^{-1/2}XX^* = (A^*A+\alpha I)^{1/2}(B^*B)^{-1/2}$$

That is,

$$(A^*A + \alpha I)^{1/2}(B^*B + \alpha I)^{-1/2}(B^*B + \alpha I)^{1/2}(B^*B)^{-1/2}$$

Thus, $A$ and $B$ are $\alpha$-almost similar.

*Remark* 3.1. The converse of Proposition 4.6 is not true in general. Consider the following example: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$

be operators on the two-dimensional Hilbert space $\mathbb{C}^2$. Define the invertible operator on $\mathbb{C}^2$ as

$$X = X^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and take $\alpha = 2$. Then $A \sim_2 B$, as in Example 1.2. Also, $A = A^*$. But $B \neq B^*$, and $B^* \neq B$.

# ORCID

0000-0002-0923-3447 (Beatrice Obiero Adhiambo)
0000-0002-9207-5573 (Victor Wanjala)

# Abbreviations

| | |
|---|---|
| $\mathfrak{B}(\mathfrak{H})$ | The set of all bounded linear operators on a Hilbert space $\mathfrak{H}$ |
| $\sigma(\mathfrak{T})$ | The spectrum of the operator $\mathfrak{T}$ |
| $\sigma_p(\mathfrak{T})$ | The point spectrum of the operator $\mathfrak{T}$ |

| | |
|---|---|
| $\sigma_{ap}(\mathfrak{T})$ | The approximate point spectrum of the operator $\mathfrak{T}$ |
| $\mathcal{T}_{(\alpha,\beta)}$ | The class of $(\alpha, \beta)$-$\mathfrak{T}$ operators |
| $\mathfrak{I}$ | The identity operator |
| $\mathbb{C}$ | The set of complex numbers |
| $\mathbb{R}$ | The set of real numbers |


## Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable remarks.


## Author Contributions

**Beatrice Obiero Adhiambo:** Conceptualization, Investigation, Methodology, Writing – original draft
**Victor Wanjala:** Formal Analysis, Validation, Writing – review & editing, Supervision

## Conflicts of Interest

The authors declare no conflicts of interest.

## References


[1] Muhammed H. Mortad, Yet more versions of the Fuglede-Putnam theorem, *Glasgow Math. J.* 51 (2009), 473-480. https://doi.org/10.1017/S001708950900518X

[2] Dehimi, S., Operators similar to their adjoints, D. Sc. Thesis, University of Oran1 Ahmed Ben Bella, (2017).

[3] Isaiah, N. S., Sammi, W. M., B. M. Nzimbi and Kikete W. D., A note on quasi-similarity in Hilbert spaces, *International Journal of Math. Archive-6(7)* (2015), 49-55.

[4] Berberian, S. K., *Introduction to Hilbert space*, Oxford university Press, New York, (1961).

[5] I. N. Sitati, B. M. Nzimbi, Stephen L. and Jairus K., Remarks on A-skew-adjoint, A-almost similarity equivalence and other operators in Hilbert space, *Pure and Applied Mathematics Journal*, 6(3) (2017), 101-107. https://doi.org/10.11648/j.pamj.20170603.12

[6] Kipkemoi, T. S., On almost similarity and other related equivalence relations of operators in Hilbert spaces, M. Sc. Thesis, University of Nairobi, (2016).

[7] Jibril, A. A., On almost similar operators, *Arabian J. Sci. Engrg.*, 21 (1996), 434-449.

[8] Campbell, S. L. and Gellar, R., Linear operators for which $T^*T$ and $TT^*$ commute, *Trans. Of the Amer. Math. Soc.*, 226 (1977), 305-319. https://doi.org/10.2307/1997907

[9] Kreyszig, E., *Introductory functional analysis with application*, Wiley, New York, (1978).

[10] Musundi, S. W., Sitati, N. I., Nzimbi, B. M and Murwayi, A. L., On almost similarity operator equivalence relation, *IJERRAS* 15(3) (2013), 293-299.

[11] Conway, J. B., *A Course in Functional Analysis*, 2nd ed., Springer-Verlag, New York, (1990). https://doi.org/10.1007/978-1-4757-3828-5

[12] Davidson, K. R., *$C^*$-Algebras by Example*, Fields Institute Monographs, Vol. 6, American Mathematical Society, Providence, RI, (1996). https://doi.org/10.1090/fim/006

[13] Fillmore, P. A., On similarity and the diagonal of a matrix, *Linear Algebra and its Applications*, 6 (1973), 177-186. https://doi.org/10.1016/0024-3795(73)90017-3

[14] Halmos, P. R., *A Hilbert Space Problem Book*, 2nd ed., Springer-Verlag, New York, (1982). https://doi.org/10.1007/978-1-4684-9330-6

[15] Simon, B., *Orthogonal Polynomials on the Unit Circle: Part 1: Classical Theory*, American Mathematical Society Colloquium Publications, Vol. 54, Providence, RI, (2005). https://doi.org/10.1090/coll/054.1

[16] Weidmann, J., *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York, (1980). https://doi.org/10.1007/978-1-4612-6027-1