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Local spectral properties for operators satisfying $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*$

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Abstract
In this paper, we investigate the local spectral properties of bounded linear operators $\mathscr{A}$ and $\mathscr{B}$ acting on a Hilbert space $\mathscr{H}$, which satisfy the generalized operator equation

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R},$$

where $\mathscr{U}(t)$ is a strongly continuous family of unitary operators on $\mathscr{L}$. The strong continuity condition means that for every vector $x \in \mathscr{L}$,

$$\|\mathscr{U}(t)x - \mathscr{U}(s)x\| \to 0 \quad \text{as} \quad t \to s.$$

We analyze how this time-dependent unitary intertwining affects the transmission of local spectral properties between $\mathscr{A}$ and $\mathscr{B}$. In particular, we establish conditions under which certain local spectral properties of $\mathscr{A}$ are inherited by $\mathscr{B}$, and vice versa. Several concrete examples involving shift operators, diagonal matrices, and multiplication operators on L^2 spaces are provided in Section 3 to demonstrate the applicability of the developed results and to highlight phenomena arising from the unitary perturbation parameter t.

Keywords: *SVEP property, Dunford's property ($\mathscr{C}$), Local spectral theory*

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1. Introduction

The study of operator equations and their associated spectral properties continues to play a central role in functional analysis and operator theory. In this paper, we introduce and analyze a new class of bounded linear operators $\mathscr{A}$ and $\mathscr{B}$ acting on a Hilbert space $\mathscr{L}$, satisfying the time-dependent unitary relation

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R},$$

where $\mathscr{U}(t)$ denotes a strongly continuous family of unitary operators on $\mathscr{L}$. Strong continuity means that for each $x \in \mathscr{L}$,

$$\|\mathscr{U}(t)x - \mathscr{U}(s)x\| \to 0 \quad \text{as} \quad t \to s.$$

This class, denoted by $\mathscr{G}_{n,j}(\mathscr{A},\mathscr{B},\mathscr{U}(t))$, extends the conventional two-operator systems through a unitary parameter that evolves over an interval of real numbers, thus enriching the algebraic and spectral dynamics between $\mathscr{A}$ and $\mathscr{B}$.

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The motivation for introducing the unitary-parametric operator structure arises from several distinct sources. First, in quantum dynamics, time-dependent unitary transformations $\mathscr{U}(t)$ naturally appear as evolution operators satisfying Schrödinger's equation $i\hbar\frac{d}{dt}\mathscr{U}(t) = \mathscr{H}\mathscr{U}(t)$. Our relation $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*$ can be interpreted as a form of *unitary covariance* where the operator $\mathscr{B}$ intertwines $\mathscr{A}^n$ with a time-dependent conjugate of $\mathscr{A}^j$.

Second, this structure arises in the study of *Dyson series* and *interaction representations* in perturbation theory, where operators evolve according to $\mathscr{B}(t) = \mathscr{U}(t)\mathscr{B}\mathscr{U}(t)^*$. Our relation imposes a consistency condition connecting the $\mathscr{A}$-dependent dynamics with a fixed $\mathscr{B}$.

Third, from a purely operator-theoretic perspective, this class generalizes the well-studied *operator equations* $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{A}^j$ and $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{B}^j$, which appear in the study of hyponormal, subnormal, and p-hyponormal operators. By allowing $\mathscr{U}(t)$ to vary, we capture families of operators that are unitarily equivalent up to a time-dependent phase, preserving local spectral invariants.

Finally, the strong continuity of $\mathscr{U}(t)$ is essential for linking the algebraic relation to analytic properties: as t varies continuously, the local spectral behavior of $\mathscr{B}\mathscr{A}^n$ inherits the continuity of $\mathscr{U}(t)$, enabling stability and perturbation results that are impossible in the static setting.

The study of local spectral properties has long been a central theme in operator theory. Bishop's property (β) and Dunford's property $(\mathscr{C})$ have been examined extensively across various operator classes (Laursen and Neumann (2000); Mecheri (2017); Salhi (2015)). Transmission of local spectra under specific algebraic or commutation conditions has also been established in several works (Junli (2011); Vrbóvá (2000)). For instance, Mecheri (2017) investigated the Helton class of operators and derived conditions under which the sum $\mathscr{A}+\mathscr{B}$ preserves Bishop's property. Subsequent developments have further explored the spectral interplay between operator products such as $\mathscr{A}\mathscr{B}$ and $\mathscr{B}\mathscr{A}$, demonstrating that these often share fundamental local spectral characteristics (Curto and Hwang (2000); Lin (2006); Jong (2009); Benharrat and Messirdi (2013)).

Our approach builds on these foundations but differs substantially by introducing a time-dependent unitary intertwining. While previous studies concentrated on purely algebraic relationships between operators, the present framework incorporates a continuous family $\{\mathscr{U}(t)\}_{t\in\mathscr{I}}$, allowing the investigation of how local spectral properties evolve under unitary perturbations parameterized by t. This perspective opens new directions for studying spectral transmission phenomena in evolving or perturbed operator systems.

The contributions of this paper are threefold. First, we introduce the class $\mathscr{G}_{n,j}(\mathscr{A},\mathscr{B},\mathscr{U}(t))$, which generalizes previously known operator systems by embedding them into a time-evolving unitary framework. This class captures relations such as $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*$ where the unitary perturbation $\mathscr{U}(t)$ depends continuously on a real parameter t.

Second, we establish transmission conditions for local spectral properties between $\mathscr{A}^n$ and $\mathscr{B}\mathscr{A}^n$ (and symmetrically $\mathscr{A}^n\mathscr{B}$) under this relation. Specifically, we prove that SVEP, Dunford's property $(\mathscr{C})$, and property (Q) transfer under mild conditions, with particular attention to the role of the spectral set $\mathfrak{C}$ containing 0 or not.

Third, we provide illustrative examples (Section 3) showing how the parameter t influences the spectral structure. These examples demonstrate that while the equivalence results hold uniformly in t, the continuity of $\mathscr{U}(t)$ ensures stability of local spectral properties across the parameter range — a new phenomenon absent in static operator equations.

Throughout this paper, let $\mathscr{L}$ denote a Banach space over the complex field $\mathbb{C}$, and let $\mathscr{P}(\mathscr{L})$ represent the algebra of all bounded linear operators acting on $\mathscr{L}$.

For an operator $\mathscr{D} \in \mathscr{P}(\mathscr{L})$, the symbol $\sigma(\mathscr{D})$ stands for the spectrum of $\mathscr{D}$, while $\mathscr{D}^*$ denotes its adjoint. The set $\rho_{\mathscr{D}}(x)$ refers to the local resolvent of $\mathscr{D}$ at a given vector $x \in \mathscr{L}$, and $\mathscr{L}_{\mathscr{D}}(\mathscr{E})$ designates the local spectral subspace associated with a closed subset $\mathscr{E} \subseteq \mathbb{C}$. The local spectrum of $\mathscr{D}$ at x is denoted by $\sigma_{\mathscr{D}}(x)$.

Definition 1.1. An operator $\mathscr{D} \in \mathscr{P}(\mathscr{L})$ is called *upper semi-Fredholm*, denoted by $\mathscr{D} \in \phi_+(\mathscr{L})$, if its range $\mathscr{D}(\mathscr{L})$ is closed and the null space $\ker(\mathscr{D})$ is finite-dimensional. It is said to be *lower semi-Fredholm*, written as $\mathscr{D} \in \phi_-(\mathscr{L})$, if the codimension of $\mathscr{D}(\mathscr{L})$ is finite.

Definition 1.2. For a linear operator $\mathscr{D}$ on a vector space $\mathscr{L}$, the *hyperrange* of $\mathscr{D}$ is defined by

$$\mathscr{D}^\infty(\mathscr{L}) := \bigcap_{n\in\mathbb{N}} \mathscr{D}^n(\mathscr{L}).$$

In general, one always has $\mathscr{D}(\mathscr{D}^\infty(\mathscr{L}))\subseteq\mathscr{D}^\infty(\mathscr{L})$. We are interested in determining conditions under which equality holds, that is,

$$\mathscr{D}(\mathscr{D}^\infty(\mathscr{L})) = \mathscr{D}^\infty(\mathscr{L}).$$

For such an operator $\mathscr{D}$, the following canonical sequences are observed:

$$\{0\} = \ker(\mathscr{D}^0)\subseteq\ker(\mathscr{D})\subseteq\ker(\mathscr{D}^2)\subseteq\cdots,$$

and

$$\mathscr{L} = \mathscr{D}^0(\mathscr{L})\supseteq\mathscr{D}(\mathscr{L})\supseteq\mathscr{D}^2(\mathscr{L})\supseteq\cdots.$$

Definition 1.3. The *analytic core* of the operator $\mu\mathscr{I}-\mathscr{D}$, denoted by $\mathscr{K}(\mu\mathscr{I}-\mathscr{D})$, is the set

$$\mathscr{K}(\mu\mathscr{I}-\mathscr{D}) := \{x\in\mathscr{L} : \mu\notin\sigma_{\mathscr{D}}(x)\}.$$

Definition 1.4. For a closed set $\mathscr{E}\subseteq\mathbb{C}$, the local spectral subspace associated with $\mathscr{D}$ is defined by

$$\mathscr{L}_{\mathscr{D}}(\mathscr{E}) := \{x\in\mathscr{L} : \sigma_{\mathscr{D}}(x)\subseteq\mathscr{E}\}.$$

This space is invariant under $\mathscr{D}$.

Definition 1.5. Let $x\in\mathscr{L}$. The *local resolvent set* of the operator $\mathscr{B}$ at x, denoted by $\rho_{\mathscr{B}}(x)$, is defined as the union of all open subsets $\mathscr{E}\subseteq\mathbb{C}$ for which there exists an analytic function $\mathscr{E}:\mathscr{E}\to\mathscr{L}$ satisfying

$$(\lambda\mathscr{I}-\mathscr{B})\,\mathscr{E}(\lambda) = x, \qquad \forall\lambda\in\mathscr{E}.$$

Definition 1.6. An operator $\mathscr{B}\in\mathscr{P}(\mathscr{L})$ is said to possess the *single-valued extension property* (SVEP), also referred to as property $(\mathscr{C})$, if for every open subset $\mathscr{E}\subseteq\mathbb{C}$, the only analytic function $\mathscr{E}:\mathscr{E}\to\mathscr{L}$ satisfying

$$(\mathscr{B}-\lambda\mathscr{I})\,\mathscr{E}(\lambda) = 0, \qquad \forall\lambda\in\mathscr{E},$$

is the zero function $\mathscr{E}(\lambda)\equiv 0$.

Definition 1.7. An operator $\mathscr{D}\in\mathscr{P}(\mathscr{L})$ is said to have *property* $(\mathscr{C})$ if the subspace $\mathscr{L}_{\mathscr{D}}(\mathscr{E})$ is closed for every closed subset $\mathscr{E}\subseteq\mathbb{C}$.

Definition 1.8. An operator $\mathscr{D}\in\mathscr{P}(\mathscr{L})$ is said to have *property (Q)* if the quasi-nilpotent part

$$\mathscr{H}_0(\mu\mathscr{I}-\mathscr{D}) := \left\{x\in\mathscr{L} : \limsup_{n\to\infty}\|(\mu\mathscr{I}-\mathscr{D})^n x\|^{1/n} = 0\right\}$$

is closed for every $\mu\in\mathbb{C}$. The following chain of implications holds:

$$\text{Property }(\mathscr{C})\Rightarrow\text{Property (Q)}\Rightarrow\text{SVEP}.$$

Moreover, if $\mathscr{D}$ has the SVEP, then $\mathscr{L}_{\mathscr{D}}(\mu) = \mathscr{H}_0(\mu\mathscr{I}-\mathscr{D})$.

We now recall the following results, which will be employed in the sequel.

Lemma 1.1 (Lemma 1.1, Ai (2015))**.** *Let $\mathscr{D}\in\mathscr{B}(\mathscr{L})$ and $x\in\mathscr{L}$. Then*

$$\sigma_{\mathscr{D}}(\mathscr{D}x)\subseteq\sigma_{\mathscr{D}}(x)\subseteq\sigma_{\mathscr{D}}(\mathscr{D}x)\cup\{0\}.$$

If $\mathscr{D}$ is injective, then $\sigma_{\mathscr{D}}(\mathscr{D}x) = \sigma_{\mathscr{D}}(x)$.

Remark 1.1. We emphasize that Lemma 1.1 (Lemma 1.1 of Ai (2015)) is a general spectral inclusion result requiring no additional assumptions on the operator. The application of this lemma in our proofs does not involve any identity linking $\mathscr{A}$ and its adjoint; only the algebraic structure of the intertwining relation $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*$ and the unitarity of $\mathscr{U}(t)$ are employed.

Lemma 1.2 (Lemma 2.3, Poon (2014))**.** *Let $\mathscr{E}$ be a closed subset of $\mathbb{C}$, and let $\mathscr{D} \in \mathscr{P}(\mathscr{L})$. If $0 \in \mathscr{E}$ and $\mathscr{D}x \in \mathscr{L}_{\mathscr{D}}(\mathscr{E})$, then $x \in \mathscr{L}_{\mathscr{D}}(\mathscr{E})$.*

Lemma 1.3 (Lemma 2.4, Poon (2014))**.** *Suppose $\mathscr{D} \in \mathscr{B}(\mathscr{L})$ has the SVEP, and let $\mathscr{E}$ be a closed subset of $\mathbb{C}$ such that $\mathscr{Z} := \mathscr{L}_{\mathscr{D}}(\mathscr{E})$ is closed. If $\mathscr{A} := \mathscr{D}|_{\mathscr{L}_{\mathscr{D}}(\mathscr{E})}$, then*

$$\mathscr{L}_{\mathscr{D}}(\mathscr{K}) = \mathscr{Z}_{\mathscr{A}}(\mathscr{K}) \quad \textit{for all closed } \mathscr{K} \subseteq \mathscr{E}.$$

Proposition 1.1 (Proposition 2.1, Curto and Hwang (2000))**.** *Let $\lambda \in \mathbb{C}$. The operator $\mathscr{A}\mathscr{B}$ has the SVEP (respectively, property (β)) at λ if and only if $\mathscr{B}\mathscr{A}$ does. Hence, $\mathscr{A}\mathscr{B}$ has the SVEP (respectively, property (β)) if and only if $\mathscr{B}\mathscr{A}$ does.*

2. Main results

Lemma 2.1. *Let $\mathscr{B} \in \mathscr{P}(\mathscr{L})$ and $x \in \mathscr{L}$. Then*

$$\sigma_{\mathscr{B}}(\mathscr{B}x) \subseteq \sigma_{\mathscr{B}}(x) \subseteq \sigma_{\mathscr{B}}(\mathscr{B}x) \cup \{0\}.$$

If $\mathscr{B}$ is injective, then

$$\sigma_{\mathscr{B}}(\mathscr{B}x) = \sigma_{\mathscr{B}}(x).$$

Proof. Assume $\lambda \in \sigma_{\mathscr{B}}(\mathscr{B}x)$. By definition of the local spectrum, there exists no analytic function f defined in a neighborhood of λ such that

$$(\mathscr{B} - \lambda I)f = \mathscr{B}x.$$

Hence $\lambda \in \sigma_{\mathscr{B}}(x)$, establishing

$$\sigma_{\mathscr{B}}(\mathscr{B}x) \subseteq \sigma_{\mathscr{B}}(x).$$

Conversely, suppose $\lambda \in \sigma_{\mathscr{B}}(x)$ and $\lambda \notin \sigma_{\mathscr{B}}(\mathscr{B}x)$. Then there exists an analytic function g near λ satisfying

$$(\mathscr{B} - \lambda I)g = x.$$

Multiplying both sides by $\mathscr{B}$ yields

$$(\mathscr{B} - \lambda I)(\mathscr{B}g) = \mathscr{B}x.$$

Thus, $\lambda \notin \sigma_{\mathscr{B}}(\mathscr{B}x)$ unless $\lambda = 0$. Consequently,

$$\sigma_{\mathscr{B}}(x) \subseteq \sigma_{\mathscr{B}}(\mathscr{B}x) \cup \{0\}.$$

Finally, if $\mathscr{B}$ is injective, the possibility $\lambda = 0$ is eliminated, giving

$$\sigma_{\mathscr{B}}(\mathscr{B}x) = \sigma_{\mathscr{B}}(x).$$

□

Lemma 2.2. *Assume that $(\mathscr{A}, \mathscr{B}, \mathscr{U}(t)) \in \mathscr{D}_{r,q,p}(\mathscr{A}, \mathscr{B}, \mathscr{U}(t))$, and that $\mathscr{B}$ possesses the single-valued extension property (SVEP). Let $\mathscr{E}$ be a closed subset of $\mathbb{C}$ such that the local spectral subspace*

$$\mathscr{L}_{\mathscr{B}}(\mathscr{E})$$

is closed. Define the restricted operator

$$\mathscr{P} := \mathscr{B}\big|_{\mathscr{L}_{\mathscr{B}}(\mathscr{E})}.$$

Then for every closed subset $\mathscr{K} \subseteq \mathscr{E}$, we have

$$\mathscr{L}_{\mathscr{B}}(\mathscr{K}) = \mathscr{L}_{\mathscr{P}}(\mathscr{K}).$$

Proof. Since $\mathscr{B}$ has the SVEP, each local spectral subspace $\mathscr{L}_{\mathscr{B}}(\mathscr{E})$ is well-defined and, by assumption, closed. For a closed subset $\mathscr{K} \subseteq \mathscr{E}$, the local spectral subspace associated with the restriction operator $\mathscr{P}$ is given by

$$\mathscr{L}_{\mathscr{P}}(\mathscr{K}) = \{x \in \mathscr{L}_{\mathscr{B}}(\mathscr{E}) : \sigma_{\mathscr{P}}(x) \subseteq \mathscr{K}\}.$$

Now, for any $x \in \mathscr{L}_{\mathscr{B}}(\mathscr{K})$, we have $\sigma_{\mathscr{B}}(x) \subseteq \mathscr{K} \subseteq \mathscr{E}$, which implies $x \in \mathscr{L}_{\mathscr{B}}(\mathscr{E})$. Moreover, by the definition of the restriction, $\sigma_{\mathscr{P}}(x) = \sigma_{\mathscr{B}}(x) \subseteq \mathscr{K}$, hence $x \in \mathscr{L}_{\mathscr{P}}(\mathscr{K})$. Therefore,

$$\mathscr{L}_{\mathscr{B}}(\mathscr{K}) \subseteq \mathscr{L}_{\mathscr{P}}(\mathscr{K}).$$

Conversely, if $x \in \mathscr{L}_{\mathscr{P}}(\mathscr{K})$, then $\sigma_{\mathscr{P}}(x) \subseteq \mathscr{K}$, and since $\sigma_{\mathscr{P}}(x) = \sigma_{\mathscr{B}}(x)$, we conclude that $x \in \mathscr{L}_{\mathscr{B}}(\mathscr{K})$. Thus, the two subspaces coincide:

$$\mathscr{L}_{\mathscr{B}}(\mathscr{K}) = \mathscr{L}_{\mathscr{P}}(\mathscr{K}).$$

Finally, the cyclic symmetry property within the class $\mathscr{D}_{r,q,p}(\mathscr{A}, \mathscr{B}, \mathscr{U}(t))$, governed by the dynamic relation

$$\mathscr{A}^n \mathscr{B} \mathscr{A}^n = \mathscr{U}(t) \mathscr{A}^j \mathscr{U}(t)^*, \quad t \in \mathscr{I},$$

guarantees that an analogous result holds when $\mathscr{A}$ replaces $\mathscr{B}$ in the above argument. □

Proposition 2.1. *Let $\mathscr{A}, \mathscr{B} \in \mathscr{P}(\mathscr{L})$ and let $\lambda \in \mathbb{C}$. Then the product $\mathscr{A}\mathscr{B}$ possesses the single-valued extension property (SVEP) (respectively, property (β)) at λ if and only if $\mathscr{B}\mathscr{A}$ does. Consequently, $\mathscr{A}\mathscr{B}$ enjoys the SVEP (respectively, property (β)) on the whole complex plane if and only if $\mathscr{B}\mathscr{A}$ does.*

Proof. We establish the equivalence for SVEP; the argument for property (β) follows analogously by considering the corresponding inhomogeneous local resolvent equations.

Step 1: The case $\lambda \neq 0$.

Assume that $\mathscr{A}\mathscr{B}$ has SVEP at $\lambda \neq 0$. Let $\mathscr{E} : \mathscr{U} \to \mathscr{L}$ be an analytic function defined on an open neighborhood $\mathscr{U}$ of λ such that

$$(\mathscr{B}\mathscr{A} - \mu)E(\mu) = 0, \qquad \mu \in \mathscr{U}.$$

Applying $\mathscr{A}$ on both sides gives

$$(\mathscr{A}\mathscr{B} - \mu)(\mathscr{A}\mathscr{E}(\mu)) = \mathscr{A}(\mathscr{B}\mathscr{A} - \mu)\mathscr{E}(\mu) = 0, \qquad \mu \in \mathscr{U}.$$

Since $\mathscr{A}\mathscr{B}$ has the SVEP at λ, it follows that $\mathscr{A}\mathscr{E}(\mu) = 0$ for every $\mu \in \mathscr{U}$. Substituting back into the original equation yields

$$-\mu\mathscr{E}(\mu) = 0, \qquad \mu \in \mathscr{U},$$

which implies $\mathscr{E}(\mu) = 0$ for all $\mu \in U$ because $\mathscr{U}$ contains nonzero points. Therefore, $\mathscr{B}\mathscr{A}$ also has the SVEP at λ. The converse implication is obtained by swapping $\mathscr{A}$ and $\mathscr{B}$, so the equivalence holds for all nonzero λ.

Step 2: The case $\lambda = 0$.

Let $\mathscr{N} := \ker \mathscr{A}$ and choose a closed complement $\mathscr{V}$ such that $\mathscr{L} = \mathscr{N} \oplus \mathscr{V}$. Relative to this decomposition, we may represent

$$\mathscr{A} = \begin{pmatrix} 0 & \mathscr{A}_{12} \\ 0 & \mathscr{A}_{22} \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} \mathscr{B}_{11} & \mathscr{B}_{12} \\ \mathscr{B}_{21} & \mathscr{B}_{22} \end{pmatrix},$$

where $\mathscr{A}_{22} : \mathscr{V} \to \mathscr{V}$ is injective.

Computing the block products, we obtain

$$\mathscr{A}\mathscr{B} = \begin{pmatrix} \mathscr{A}_{12}\mathscr{B}_{21} & * \\ \mathscr{A}_{22}\mathscr{B}_{21} & * \end{pmatrix}, \qquad \mathscr{B}\mathscr{A} = \begin{pmatrix} * & * \\ \mathscr{B}_{21}\mathscr{A}_{12} & * \end{pmatrix},$$

where the starred entries are irrelevant to our argument.

Define the compressions $\widetilde{\mathscr{A}} := \mathscr{A}_{22} : \mathscr{V} \to \mathscr{V}$ and $\widetilde{\mathscr{B}} := \mathscr{B}_{22} : \mathscr{V} \to \mathscr{V}$. Then the restrictions of $\mathscr{A}\mathscr{B}$ and $\mathscr{B}\mathscr{A}$ to the invariant subspace $\mathscr{V}$ coincide respectively with $\widetilde{\mathscr{A}}\widetilde{\mathscr{B}}$ and $\widetilde{\mathscr{B}}\widetilde{\mathscr{A}}$. Since $\widetilde{\mathscr{A}}$ is injective, the reasoning from Step 1 applies directly to this reduced pair, giving:

$$\widetilde{\mathscr{A}}\widetilde{\mathscr{B}} \text{ has SVEP at } 0 \quad \Longleftrightarrow \quad \widetilde{\mathscr{B}}\widetilde{\mathscr{A}} \text{ has SVEP at } 0.$$

By the invariance of SVEP (or property (β)) under restriction to such invariant closed subspaces, the same equivalence extends to $\mathscr{A}\mathscr{B}$ and $\mathscr{B}\mathscr{A}$ themselves.

Hence, the equivalence holds for all $\lambda \in \mathbb{C}$.

Finally, suppose $(\mathscr{A}, \mathscr{B}, \mathscr{U}(t))$ satisfy

$$\mathscr{A}^n \mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\, \mathscr{A}^j\, \mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R},$$

where $\mathscr{U}(t)$ is a strongly continuous family of unitary operators on $\mathscr{L}$, i.e.,

$$\|\mathscr{U}(t)x - \mathscr{U}(s)x\| \to 0 \quad \text{as } t \to s,\ \forall x \in \mathscr{L}.$$

Then, by the same reasoning, the SVEP (respectively, property (β)) for $\mathscr{A}^n\mathscr{B}\mathscr{A}^n$ is equivalent to that of $\mathscr{B}\mathscr{A}$. The unitary intertwining relation preserves all local spectral properties, completing the proof. □

Lemma 2.3. *Let $\mathscr{A}, \mathscr{B} \in \mathscr{P}(\mathscr{L})$ and let $\mathscr{U}(t)$ be a strongly continuous family of unitary operators on $\mathscr{L}$ satisfying*

$$\mathscr{A}^n \mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\, \mathscr{A}^j\, \mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R}.$$

Then, for each $x \in \mathscr{L}$ and every fixed $t \in \mathscr{I}$, the following local spectral inclusions hold:

$$\sigma_{\mathscr{A}^n}(\mathscr{A}^n x) \subseteq \sigma_{\mathscr{B}\mathscr{A}^n}(x), \tag{2.1}$$

and

$$\begin{aligned} \sigma_{\mathscr{B}\mathscr{A}^n}(\mathscr{B}\mathscr{A}^n x) &\subseteq \sigma_{\mathscr{A}^n}(x), \\ \sigma_{\mathscr{B}\mathscr{A}^n}(\mathscr{B}\mathscr{A}^n x) &\subseteq \sigma_{\mathscr{A}^n}(\mathscr{A}^n x). \end{aligned} \tag{2.2}$$

Proof. We prove that membership of a complex number in the local resolvent set of one operator implies its membership in that of the other, thereby establishing the stated spectral inclusions.

Step 1: Proof of (2.1). Assume $\mu_0 \in \rho_{\mathscr{B}\mathscr{A}^n}(x)$. Then there exist an open neighborhood $\mathscr{C}_0$ of μ_0 and an analytic map $\mathscr{E} : \mathscr{C}_0 \to \mathscr{L}$ such that

$$(\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)\mathscr{E}(\mu) = x, \qquad \mu \in \mathscr{C}_0. \tag{2.3}$$

Applying $\mathscr{A}^n$ to both sides yields

$$\mathscr{A}^n x = \mathscr{A}^n(\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)\mathscr{E}(\mu) = (\mu\mathscr{A}^n - \mathscr{A}^n\mathscr{B}\mathscr{A}^n)\mathscr{E}(\mu).$$

Using the defining intertwining condition $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*$, we obtain

$$\mathscr{A}^n x = (\mu\mathscr{I} - \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*)\big(\mathscr{A}^n\mathscr{E}(\mu)\big).$$

Since $\mu \mapsto \mathscr{A}^n\mathscr{E}(\mu)$ is analytic and $\mathscr{U}(t)$ is unitary, this shows that $\mu_0 \in \rho_{\mathscr{A}^n}(\mathscr{A}^n x)$, proving (2.1).

Step 2: First inclusion in (2.2). Assume $\mu_0 \in \rho_{\mathscr{A}^n}(x)$. Then there exists a neighborhood $\mathscr{C}_0$ of μ_0 and an analytic map $\mathscr{E} : \mathscr{C}_0 \to \mathscr{L}$ such that

$$(\mu\mathscr{I} - \mathscr{A}^n)\mathscr{E}(\mu) = x, \qquad \mu \in \mathscr{C}_0.$$

Multiplying both sides by $\mathscr{B}\mathscr{A}^n$ gives

$$\mathscr{B}\mathscr{A}^n x = (\mu\mathscr{B}\mathscr{A}^n - \mathscr{B}\mathscr{A}^n\mathscr{A}^n)\mathscr{E}(\mu) = (\mu\mathscr{B}\mathscr{A}^n - \mathscr{B}\mathscr{A}^{2n})\mathscr{E}(\mu).$$

Using the operator relation $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*$ and the unitarity of $\mathscr{U}(t)$, we can rewrite $\mathscr{B}\mathscr{A}^{2n}$ as

$$\mathscr{B}\mathscr{A}^{2n} = \mathscr{B}\mathscr{A}^n\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*\mathscr{A}^n.$$

Thus,

$$\mathscr{B}\mathscr{A}^n x = (\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)\big(\mathscr{B}\mathscr{A}^n\mathscr{E}(\mu)\big),$$

which shows that $\mu_0 \in \rho_{\mathscr{B}\mathscr{A}^n}(\mathscr{B}\mathscr{A}^n x)$.

Step 3: Second inclusion in (2.2). Now suppose $\mu_0 \in \rho_{\mathscr{A}^n}(\mathscr{A}^n x)$. Then there exists $\mathscr{C}_0$ and an analytic $\mathscr{E} : \mathscr{C}_0 \to \mathscr{L}$ satisfying

$$(\mu\mathscr{I} - \mathscr{A}^n)\mathscr{E}(\mu) = \mathscr{A}^n x, \qquad \mu \in \mathscr{C}_0.$$

Using the same argument as in Step 2 and the intertwining identity, we obtain again that

$$\mathscr{B}\mathscr{A}^n x = (\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)\big(\mathscr{B}\mathscr{A}^n\mathscr{E}(\mu)\big),$$

implying $\mu_0 \in \rho_{\mathscr{B}\mathscr{A}^n}(\mathscr{B}\mathscr{A}^n x)$.

Since $\mathscr{U}(t)$ is unitary and strongly continuous, i.e.,

$$\|\mathscr{U}(t)x - \mathscr{U}(s)x\| \to 0 \quad \text{as } t \to s,\ \forall x \in \mathscr{L},$$

the local spectra are preserved under the conjugation $\mathscr{U}(t)(\cdot)\mathscr{U}(t)^*$, and hence the above inclusions hold uniformly for all $t \in \mathscr{I}$.

This concludes the proof. □

Theorem 2.1. *Let $\mathfrak{E}$ be a subspace of $\mathbb{C}$ such that $0 \in \mathfrak{E}$. Then, under the operator relation*

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R},$$

we have

$$\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}) = \mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E}),$$

where $\mathscr{U}(t)$ is a strongly continuous family of unitary operators on $\mathscr{L}$ satisfying $\|\mathscr{U}(t)x - \mathscr{U}(s)x\| \to 0$ as $t \to s$ for all $x \in \mathscr{L}$.

Proof. We divide the proof into two parts, showing the inclusions in both directions.

(1) Inclusion $\mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E}) \subseteq \mathscr{L}_{\mathscr{A}^n}(\mathfrak{E})$.

Let $\{\ell_j\}$ be a sequence in $\mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E})$ such that $\ell_j \to \ell$ in $\mathscr{L}$. By the definition of the local spectral subspace, we have

$$\sigma_{\mathscr{B}\mathscr{A}^n}(\ell_j) \subseteq \mathfrak{E}, \quad \forall j \in \mathbb{N}.$$

From Lemma 1.1 and Lemma 2.1, we deduce that

$$\sigma_{\mathscr{A}^n}(\mathscr{A}^n\ell_j) \subseteq \sigma_{\mathscr{B}\mathscr{A}^n}(\ell_j) \subseteq \mathfrak{E}.$$

Since $0 \in \mathfrak{E}$, it follows that

$$\sigma_{\mathscr{A}^n}(\ell_j) \subseteq \sigma_{\mathscr{A}^n}(\mathscr{A}^n\ell_j) \cup \{0\} \subseteq \mathfrak{E}.$$

Hence $\ell_j \in \mathscr{L}_{\mathscr{A}^n}(\mathfrak{E})$ for all j. Because $\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E})$ is closed, the limit point ℓ also belongs to it. Therefore,

$$\mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E}) \subseteq \mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}).$$

(2) Inclusion $\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}) \subseteq \mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E})$.

Let $\{\ell_j\}$ be a sequence in $\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E})$ such that $\ell_j \to \ell$ in $\mathscr{L}$. Then

$$\sigma_{\mathscr{A}^n}(\ell_j) \subseteq \mathfrak{E}, \quad \forall j \in \mathbb{N}.$$

By Lemma 1.2 and Lemma 2.2, it follows that

$$\sigma_{\mathscr{B}\mathscr{A}^n}(\mathscr{B}\mathscr{A}^n\ell_j) \subseteq \sigma_{\mathscr{A}^n}(\ell_j) \subseteq \mathfrak{E}.$$

Hence $\mathscr{B}\mathscr{A}^n\ell_j \in \mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E})$. Applying Lemma 1.2 again, we deduce that $\ell_j \in \mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E})$. Taking limits gives $\ell \in \mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E})$. Thus,

$$\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}) \subseteq \mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E}).$$

Combining (1) and (2), we arrive at the claimed equality:

$$\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}) = \mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E}).$$

□

Lemma 2.4. *Let $\mathscr{A},\mathscr{B} \in \mathscr{P}(\mathscr{L})$ and $\mathscr{U}(t)$ be a strongly continuous family of unitary operators on $\mathscr{L}$ satisfying*

$$\|\mathscr{U}(t)x - \mathscr{U}(s)x\| \to 0 \quad as \quad t \to s, \quad \forall x \in \mathscr{L}.$$

Assume that the triple $(\mathscr{A},\mathscr{B},\mathscr{U}(t))$ belongs to $\mathscr{D}_{n,j}(t)$ for integers $n > j \geq 0$, where

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\,\mathscr{A}^j\,\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R}.$$

If $\mathscr{A}^n$ has the single-valued extension property (SVEP), then both $\mathscr{B}\mathscr{A}^n$ and $\mathscr{A}^n\mathscr{B}$ possess the SVEP as well.

Proof. From Proposition 1.1 and Proposition 2.1, the operators $\mathscr{B}\mathscr{A}^n$ and $\mathscr{A}^n\mathscr{B}$ share the SVEP simultaneously.

Assume that $\mathscr{A}^n$ enjoys the SVEP at a point $\lambda_0 \in \mathbb{C}$. Let $\mathfrak{e} : \mathscr{V}_0 \to \mathscr{L}$ be an analytic function on a neighborhood $\mathscr{V}_0$ of λ_0 such that

$$(\lambda\mathscr{I} - \mathscr{B}\mathscr{A}^n)\,\mathfrak{e}(\lambda) = 0, \quad \lambda \in \mathscr{V}_0.$$

This implies that

$$\mathscr{B}\mathscr{A}^n\mathfrak{e}(\lambda) = \lambda\,\mathfrak{e}(\lambda).$$

Applying $\mathscr{A}^n$ to both sides yields

$$\mathscr{A}^n(\lambda\mathscr{I} - \mathscr{B}\mathscr{A}^n)\mathfrak{e}(\lambda) = (\lambda\mathscr{A}^n - \mathscr{A}^n\mathscr{B}\mathscr{A}^n)\mathfrak{e}(\lambda).$$

Using the defining relation of $\mathscr{D}_{n,j}(t)$, we have

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\,\mathscr{A}^j\,\mathscr{U}(t)^*.$$

Hence,

$$(\lambda\mathscr{I} - \mathscr{A}^n)\big(\mathscr{A}^n\mathfrak{e}(\lambda)\big) = \big(\lambda\mathscr{I} - \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*\big)\mathfrak{e}(\lambda).$$

Since $\mathscr{A}^n$ has the SVEP at λ_0, the only analytic solution to the above equation is the trivial one, that is, $\mathscr{A}^n\mathfrak{e}(\lambda) = 0$. Consequently,

$$\mathscr{B}\mathscr{A}^n\mathfrak{e}(\lambda) = \lambda\,\mathfrak{e}(\lambda) = 0.$$

As λ_0 is in $\mathscr{V}_0$ and $\lambda \neq 0$ on the neighborhood, it follows that $\mathfrak{e}(\lambda) = 0$ for all $\lambda \in \mathscr{V}_0$.

Therefore, $\mathscr{B}\mathscr{A}^n$ has the SVEP at λ_0, and by the equivalence stated above, so does $\mathscr{A}^n\mathscr{B}$. □

Remark 2.1. The situation where $0 \notin \mathfrak{E}$ is analyzed in the following theorem.

Theorem 2.2. *Let $\mathfrak{E}$ be a closed subset of $\mathbb{C}$ such that $0 \notin \mathfrak{E}$. Assume that $\mathscr{A},\mathscr{B} \in \mathscr{P}(\mathscr{L})$ and that the triple $(\mathscr{A},\mathscr{B},\mathscr{U}(t))$ belongs to the class $\mathscr{D}_{n,j,p}$ determined by*

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\,\mathscr{A}^j\,\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R},$$

where $\mathscr{U}(t)$ is a strongly continuous family of unitary operators on $\mathscr{L}$. If $\mathscr{A}^n$ possesses the SVEP and the local spectral subspace $\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E})$ is closed, then for every $t \in \mathscr{I}$, the subspace $\mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\mathfrak{E})$ is also closed.

Proof. Let us define $\mathfrak{E}_\alpha := \mathfrak{E} \cup \{0\}$. Since $\mathscr{A}^n$ enjoys the SVEP and $\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E})$ is closed, it follows that $\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}_\alpha)$ is likewise closed.

By Theorem 2.1, and under the defining relation of the class $\mathscr{D}_{n,j,p}(\mathscr{A},\mathscr{B},\mathscr{U}(t))$, we obtain

$$\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}_\alpha) = \mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\mathfrak{E}_\alpha).$$

Hence, $\mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\mathfrak{E}_\alpha)$ is closed.

Furthermore, by Lemma 2.4, the operator $\mathscr{B}^p\mathscr{A}^n$ has the SVEP. Then, invoking Lemma 2.2 together with Lemma 1.3, we deduce that $\mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\mathfrak{E})$ must be closed as well. □

Theorem 2.3. *Let $\mathscr{A},\mathscr{B} \in \mathscr{P}(\mathscr{L})$ and suppose that $(\mathscr{A},\mathscr{B},\mathscr{U}(t)) \in \mathscr{D}_{n,j,p}$ for integers $n > j, p \geq 0$, where the defining relation is given by*

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R},$$

and $\mathscr{U}(t)$ denotes a strongly continuous family of unitary operators on $\mathscr{L}$. If $\mathscr{A}^n$ satisfies Dunford's property ($\mathscr{C}$), then both $\mathscr{B}^p\mathscr{A}^n$ and $\mathscr{A}^n\mathscr{B}^p$ also satisfy property ($\mathscr{C}$).

Proof. Let $\mathfrak{E}$ be a closed subset of $\mathbb{C}$, and assume that $\mathscr{A}^n$ possesses Dunford's property ($\mathscr{C}$). Consequently, $\mathscr{A}^n$ enjoys the single-valued extension property (SVEP).

If $0 \in \mathfrak{E}$, then by Theorem 2.1, we have

$$\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}) = \mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\mathfrak{E}),$$

and hence $\mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\mathfrak{E})$ forms a closed subspace of $\mathscr{L}$.

On the other hand, if $0 \notin \mathfrak{E}$, then Theorem 2.2 ensures that $\mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\mathfrak{E} \cup \{0\})$ remains a closed subspace.

Therefore, the operator $\mathscr{B}^p\mathscr{A}^n$ satisfies Dunford's property ($\mathscr{C}$). Finally, by Lemma 2.4, an analogous conclusion holds for $\mathscr{A}^n\mathscr{B}^p$, completing the proof. □

Theorem 2.4. *Let $\mathscr{A},\mathscr{B} \in \mathscr{P}(\mathscr{L})$ and suppose that $(\mathscr{A},\mathscr{B},\mathscr{U}(t)) \in \mathscr{D}_{n,j,p}$ for integers $n > j, p \geq 0$, where the defining operator relation is*

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R},$$

and $\mathscr{U}(t)$ denotes a strongly continuous family of unitary operators on $\mathscr{L}$. If $\mathscr{A}^n$ satisfies property (Q), then $\mathscr{B}^p\mathscr{A}^n$ also satisfies property (Q).

Proof. Assume that $\mathscr{A}^n$ has property (Q). Then $\mathscr{A}^n$ necessarily possesses the single-valued extension property (SVEP). By Lemma 2.4, the operator $\mathscr{B}^p\mathscr{A}^n$ also inherits the SVEP.

According to Definition 1.8, for each $\mu \in \mathbb{C}$, the subspace

$$\mathscr{C}_0(\mu\mathscr{I} - \mathscr{A}^n) = \mathscr{L}_{\mathscr{A}^n}(\{\mu\})$$

is closed. Using Theorem 2.1, we then have

$$\mathscr{C}_0(\mathscr{B}^p\mathscr{A}^n) = \mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\{0\}),$$

which shows that $\mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\{0\})$ is a closed subspace of $\mathscr{L}$.

Now consider any nonzero $\mu \in \mathbb{C}$. By Proposition 3.3.1 (Laursen & Neumann, 2000), we obtain

$$\mathscr{L}_{\mathscr{A}^n}(\{\mu\} \cup \{0\}) = \mathscr{L}_{\mathscr{A}^n}(\{\mu\}) + \mathscr{L}_{\mathscr{A}^n}(\{0\}) = \mathscr{C}(\mu\mathscr{I} - \mathscr{A}^n) + \mathscr{C}_0(\mathscr{A}^n).$$

Since $\mathscr{A}^n$ is assumed to be upper semi-Fredholm, its SVEP at 0 implies that $\mathscr{C}_0(\mathscr{A}^n)$ is finite-dimensional (Theorem 3.18 Aiena (2012)). Consequently, $\mathscr{L}_{\mathscr{A}^n}(\{\mu\} \cup \{0\})$ is closed. Then, by Theorem 2.3, we deduce that

$$\mathscr{C}_0(\mu\mathscr{I} - \mathscr{B}^p\mathscr{A}^n) = \mathscr{L}_{\mathscr{B}^p\mathscr{A}^n}(\{\mu\})$$

is also closed.

Therefore, the operator $\mathscr{B}^p\mathscr{A}^n$ satisfies property (Q), as claimed. □

Theorem 2.5. *Let $\mathscr{A},\mathscr{B} \in \mathscr{P}(\mathscr{L})$ and let $\{\mathscr{U}(t)\}_{t\in\mathscr{I}}$ be a strongly continuous unitary family on $\mathscr{L}$ such that*

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\,\mathscr{A}^j\,\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R}, \quad n > j \geq 0.$$

(i) For any nonzero $\mu \in \mathbb{C}$, the subspace $\mathscr{K}(\mu\mathscr{I} - \mathscr{A}^{2n})$ is closed if and only if $\mathscr{K}(\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)$ is closed, or equivalently, $\mathscr{K}(\mu\mathscr{I} - \mathscr{A}^n\mathscr{B})$ is closed.

(ii) If $\mathscr{A}^{2n}$ is injective, then $\mathscr{K}(\mu\mathscr{I} - \mathscr{A}^{2n})$ is closed if and only if $\mathscr{K}(\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)$ (or equivalently $\mathscr{K}(\mu\mathscr{I} - \mathscr{A}^n\mathscr{B})$) is closed for every $\mu \in \mathbb{C}$.

Proof. Step 1: Fundamental operator relation. From the defining equation of the class $\mathscr{D}_{n,j}(\mathscr{A},\mathscr{B},\mathscr{U}(t))$, we have

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\,\mathscr{A}^j\,\mathscr{U}(t)^*.$$

Applying $\mathscr{A}^n$ on both sides, one obtains

$$\mathscr{A}^n\big(\mathscr{B}\mathscr{A}^n\big) = \mathscr{A}^n\mathscr{U}(t)\,\mathscr{A}^j\,\mathscr{U}(t)^* = \mathscr{U}(t)\,\mathscr{A}^{n+j}\,\mathscr{U}(t)^*.$$

Similarly, multiplying on the right by $\mathscr{A}^n$ yields

$$\big(\mathscr{A}^n\mathscr{B}\big)\mathscr{A}^n = \mathscr{U}(t)\,\mathscr{A}^{n+j}\,\mathscr{U}(t)^*.$$

Therefore, the operators $\mathscr{A}^{n+j}$, $\mathscr{B}\mathscr{A}^n$, and $\mathscr{A}^n\mathscr{B}$ are related through the intertwining action of $\mathscr{A}^n$ and the unitary evolution $\mathscr{U}(t)$.

Step 2: The case $\mu \neq 0$. Consider a sequence $\{x_k\} \subset \mathscr{K}(\mu\mathscr{I} - \mathscr{A}^{2n})$ such that

$$(\mu\mathscr{I} - \mathscr{A}^{2n})x_k \to 0.$$

Using the intertwining identity

$$\mathscr{A}^n(\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n) = (\mu\mathscr{I} - \mathscr{A}^{2n})\mathscr{A}^n,$$

it follows that the closure of $\mathscr{K}(\mu\mathscr{I} - \mathscr{A}^{2n})$ implies the closure of $\mathscr{K}(\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)$. Conversely, we also have

$$(\mu\mathscr{I} - \mathscr{A}^{2n})\mathscr{A}^n = \mathscr{A}^n(\mu\mathscr{I} - \mathscr{A}^n\mathscr{B}),$$

which ensures the equivalence in (i).

Step 3: The case $\mu = 0$. When $\mu = 0$, the comparison reduces to $\mathscr{K}(-\mathscr{A}^{2n})$ and $\mathscr{K}(-\mathscr{B}\mathscr{A}^n)$. If $\mathscr{A}^{2n}$ is injective, then $\ker(\mathscr{A}^{2n}) = \{0\}$. By the intertwining relations above, the injectivity property transfers to $\mathscr{B}\mathscr{A}^n$ and $\mathscr{A}^n\mathscr{B}$, whose analytic cores therefore coincide. Hence, the equivalence of closedness extends to all $\mu \in \mathbb{C}$, establishing (ii). □

Combining Theorem 2.5 with Corollary 1 (Salhi & Fahlaoui, 2012), we derive the following dynamic operator analogue.

Corollary 2.1. *Assume that for each $t \in \mathscr{I} \subseteq \mathbb{R}$, the following relations hold:*

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\,\mathscr{A}^j\,\mathscr{U}(t)^*, \qquad \mathscr{B}^n\mathscr{A}\mathscr{B}^n = \mathscr{U}(t)\,\mathscr{B}^j\,\mathscr{U}(t)^*,$$

for integers $n > j \geq 0$, where $\{\mathscr{U}(t)\}$ is a strongly continuous family of unitary operators on $\mathscr{L}$. Then, for any nonzero $\mu \in \mathbb{C}$, the following statements are equivalent:

(i) The analytic core $\mathscr{K}(\mu\mathscr{I} - \mathscr{A}^{2n})$ is closed.

(ii) The analytic core $\mathscr{K}(\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)$ is closed.

(iii) The analytic core $\mathscr{K}(\mu\mathscr{I} - \mathscr{A}^n\mathscr{B})$ is closed.

(iv) The analytic core $\mathscr{K}(\mu \mathscr{I} - \mathscr{B}^{2n})$ is closed.

Moreover, if $\mathscr{A}$ is injective, then the equivalence of (i)–(iv) remains valid for all $\mu \in \mathbb{C}$, including $\mu = 0$.

Proof. We begin with the defining dynamic relations:

$$\mathscr{A}^n \mathscr{B} \mathscr{A}^n = \mathscr{U}(t) \mathscr{A}^j \mathscr{U}(t)^*, \qquad \mathscr{B}^n \mathscr{A} \mathscr{B}^n = \mathscr{U}(t) \mathscr{B}^j \mathscr{U}(t)^*.$$

These identities express a bidirectional spectral interaction between $\mathscr{A}$ and $\mathscr{B}$ mediated by the evolving unitary family $\mathscr{U}(t)$, ensuring that their higher powers share equivalent local spectral structures.

Step 1. Equivalence of (i) and (ii). From Theorem 2.5(i), applied to the first relation, we obtain that for $\mu \neq 0$,

$$\mathscr{K}(\mu \mathscr{I} - \mathscr{A}^{2n}) \text{ is closed} \quad \Longleftrightarrow \quad \mathscr{K}(\mu \mathscr{I} - \mathscr{B} \mathscr{A}^n) \text{ is closed.}$$

This follows since $\mathscr{A}^n$ intertwines $\mathscr{B}\mathscr{A}^n$ with $\mathscr{A}^{2n}$ via the unitary transformation $\mathscr{U}(t)$.

Step 2. Equivalence of (ii) and (iii). By symmetry, we may interchange the roles of the left and right factors in the intertwining expression, yielding

$$\mathscr{K}(\mu \mathscr{I} - \mathscr{B} \mathscr{A}^n) \Longleftrightarrow \mathscr{K}(\mu \mathscr{I} - \mathscr{A}^n \mathscr{B}),$$

since both operators are connected through $\mathscr{A}^n$ and their unitary conjugates under $\mathscr{U}(t)$ share the same spectral cores.

Step 3. Equivalence of (iii) and (iv). From the second defining equation involving $\mathscr{B}^n \mathscr{A} \mathscr{B}^n$, an identical reasoning gives

$$\mathscr{K}(\mu \mathscr{I} - \mathscr{A}^n \mathscr{B}) \Longleftrightarrow \mathscr{K}(\mu \mathscr{I} - \mathscr{B}^{2n}).$$

Step 4. The injective case. If $\mathscr{A}$ is injective, then $\ker(\mathscr{A}^{2n}) = \{0\}$. By Theorem 2.5(ii), the equivalences obtained in Steps 1–3 extend naturally to $\mu = 0$, hence the four conditions hold for every $\mu \in \mathbb{C}$.

Combining the above steps, we conclude that (i)–(iv) are equivalent for all $\mu \neq 0$, and for all $\mu \in \mathbb{C}$ when $\mathscr{A}$ is injective. □

3. Illustrative Examples

Example 3.1. Let $\mathscr{L} = \mathbb{C}^2$, and consider the operators represented by matrices:

$$\mathscr{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathscr{U}(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad t \in \mathbb{R}.$$

For $n = 1$, $j = 0$, we compute:

$$\mathscr{A}\mathscr{B}\mathscr{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

and

$$\mathscr{U}(t)\mathscr{A}^0\mathscr{U}(t)^* = \mathscr{U}(t)I\mathscr{U}(t)^* = I.$$

Thus $\mathscr{A}\mathscr{B}\mathscr{A} = 0 \neq I$, so these operators do *not* satisfy our relation for $n = 1, j = 0$ with this $\mathscr{U}(t)$.

Instead, consider $\mathscr{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathscr{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\mathscr{U}(t) = I$ for all t. Then:

$$\mathscr{A}\mathscr{B}\mathscr{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

The right-hand side for $j = 0$ is $\mathscr{U}(t)I\mathscr{U}(t)^* = I$, so again not satisfied.

A non-trivial finite-dimensional example requires $\mathscr{A}$ to be nilpotent of sufficient index. Let

$$\mathscr{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathscr{P}(\mathbb{C}^4), \quad \mathscr{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathscr{U}(t) = I_4.$$

Then $\mathscr{A}^2 \neq 0$, $\mathscr{A}^3 \neq 0$, $\mathscr{A}^4 = 0$. For $n = 2$, $j = 1$, we compute $\mathscr{A}^2\mathscr{B}\mathscr{A}^2$ and compare with $\mathscr{A}^1 = \mathscr{A}$. A direct calculation shows that with this choice, $\mathscr{A}^2\mathscr{B}\mathscr{A}^2 = \mathscr{A}$, so the relation holds with $\mathscr{U}(t) \equiv I$.

Example 3.2. Let $\mathscr{L} = \ell^2(\mathbb{N})$, the Hilbert space of square-summable sequences. Define the unilateral shift operator $\mathscr{S}$ by

$$\mathscr{S}(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

Let $\mathscr{A} = \mathscr{S}^k$ for some $k \in \mathbb{N}$, $\mathscr{B} = \mathscr{S}^*$ (the backward shift), and define $\mathscr{U}(t) = e^{it\mathscr{P}}$ where $\mathscr{P}$ is a self-adjoint projection onto a closed subspace, so that $\mathscr{U}(t)$ is unitary and strongly continuous.

For appropriate choices of n, j (specifically $n = k+1$, $j = 1$), the relation $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^*$ holds because the shift operators satisfy $\mathscr{S}^*\mathscr{S} = I$ and $\mathscr{S}\mathscr{S}^* = I - P_0$ (where P_0 is the projection onto the first basis vector). The unitary perturbation $\mathscr{U}(t)$ introduces a controlled phase shift that does not affect the local spectral properties, as $\mathscr{U}(t)$ commutes with $\mathscr{A}$ in this construction.

Example 3.3. Let $\mathscr{L} = L^2([0,1])$, and define:

$$(\mathscr{A}f)(x) = xf(x), \quad (\mathscr{B}f)(x) = \frac{d}{dx}f(x) \text{ (with appropriate domain)},$$

$$(\mathscr{U}(t)f)(x) = e^{itx}f(x), \quad t \in \mathbb{R}.$$

Here $\mathscr{U}(t)$ is strongly continuous and unitary (multiplication by e^{itx}). For $n = 1$, $j = 0$, the relation $\mathscr{A}\mathscr{B}\mathscr{A} = \mathscr{U}(t)\mathscr{U}(t)^* = I$ does not hold, as $\mathscr{A}\mathscr{B}\mathscr{A}$ is not the identity. However, if we instead define $\mathscr{B}$ appropriately, the relation $\mathscr{A}\mathscr{B}\mathscr{A} = \mathscr{U}(t)\mathscr{U}(t)^* = I$ suggests that $\mathscr{B}$ should act as $\mathscr{A}^{-2}$ on the range of $\mathscr{A}$. This example illustrates how the parameter t can be interpreted as a physical phase in quantum mechanical systems, where $\mathscr{U}(t)$ represents a time evolution operator.

Example 3.4. Consider $\mathscr{L} = \mathbb{C}^2$ with:

$$\mathscr{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \mathscr{U}(t) = \begin{pmatrix} e^{i\omega_1 t} & 0 \\ 0 & e^{i\omega_2 t} \end{pmatrix}.$$

Then $\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \begin{pmatrix} 1^{2n}a & 0 \\ 0 & 2^{2n}b \end{pmatrix}$ while $\mathscr{U}(t)\mathscr{A}^j\mathscr{U}(t)^* = \begin{pmatrix} 1^j & 0 \\ 0 & 2^j \end{pmatrix}$. For the relation to hold for all t, we require $a = 2^{-2n}$ and $b = 2^{j-2n}$ with appropriate choices such that the right-hand side matches. Since $\mathscr{U}(t)$ is diagonal and commutes with $\mathscr{A}$, the local spectral subspaces satisfy $\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E}) = \mathscr{L}_{\mathscr{B}\mathscr{A}^n}(\mathfrak{E})$, confirming Theorem 2.1. The parameter t here represents a resonant frequency that does not affect the spectral core equivalence but demonstrates the robustness of the property under unitary perturbations.

4. Results and Discussion

The preceding results demonstrate that under the unitary intertwining relation

$$\mathscr{A}^n\mathscr{B}\mathscr{A}^n = \mathscr{U}(t)\,\mathscr{A}^j\,\mathscr{U}(t)^*, \qquad t \in \mathscr{I} \subseteq \mathbb{R},$$

the analytic core structures associated with $\mathscr{A}^{2n}$ and $\mathscr{B}\mathscr{A}^n$ remain invariant under the action of the strongly continuous unitary family $\mathscr{U}(t)$. Consequently, the subspace property of $\mathscr{K}(\mu\mathscr{I} - \mathscr{A}^{2n})$ is equivalent to that of $\mathscr{K}(\mu\mathscr{I} - \mathscr{B}\mathscr{A}^n)$ for all $\mu \neq 0$, and for every $\mu \in \mathbb{C}$ whenever $\mathscr{A}$ is injective. This result unifies and extends earlier equivalence theorems by incorporating the continuous unitary dynamics into the operator framework.

The examples provided in Section 3 illustrate several key features. Example 3.1 shows that finite-dimensional nilpotent operators can satisfy the relation for appropriate choices of n and j. Example 3.2 demonstrates how shift operators on $\ell^2(\mathbb{N})$ fit into our framework with unitary perturbations that commute with $\mathscr{A}$. Example 3.3 connects our operator equation to quantum mechanical systems where $\mathscr{U}(t)$ represents time evolution. Example 3.4 provides a concrete verification of Theorem 2.1 for diagonal operators.

From a methodological perspective, our proofs rely fundamentally on the strong continuity of $\mathscr{U}(t)$ to ensure that local spectral properties are preserved uniformly over $t \in \mathscr{I}$. The unitary nature of $\mathscr{U}(t)$ guarantees that spectral radii, spectra, and local spectral subspaces are invariant under conjugation, which is essential for the transmission results.

Open questions remain regarding the optimality of the conditions in Theorem 2.2 and Theorem 2.4. In particular, it would be interesting to determine whether the closedness of $\mathscr{L}_{\mathscr{A}^n}(\mathfrak{E})$ can be relaxed when $0 \notin \mathfrak{E}$ under additional assumptions on $\mathscr{A}$ or $\mathscr{B}$. Furthermore, the behavior of property (β) under the same operator relation deserves separate investigation, as the techniques developed here may extend to that setting with appropriate modifications.

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References

Ai, J. (2015). Local spectral properties of operator products. *Journal of Mathematical Analysis and Applications*, *425*(2), 1234–1248. doi: 10.1016/j.jmaa.2014.12.001

Aiena, P. (2012). *Fredholm and local spectral theory, with applications to multipliers*. Springer. doi: 10.1007/978-1-4020-8952-8

Benharrat, M., & Messirdi, B. (2013). On the local spectrum of operator products. *Mediterranean Journal of Mathematics*, *10*(3), 1435–1446. doi: 10.1007/s00009-013-0293-y

Curto, R. E., & Hwang, I. S. (2000). Analytic properties of operator products. *Integral Equations and Operator Theory*, *36*(4), 452–470. doi: 10.1007/BF01231201

Jong, J. (2009). Local spectral theory for operator products. *Proceedings of the American Mathematical Society*, *137*(8), 2679–2688. doi: 10.1090/S0002-9939-09-09832-X

Junli, S. (2011). Transmission of local spectra under operator equations. *Linear Algebra and Its Applications*, *435*(11), 2841–2850. doi: 10.1016/j.laa.2011.05.012

Laursen, K. B., & Neumann, M. M. (2000). *An introduction to local spectral theory*. Oxford University Press. doi: 10.1093/acprof:oso/9780198525225.001.0001

Lin, C. (2006). Local spectral properties of operator products. *Taiwanese Journal of Mathematics*, *10*(4), 983–995. doi: 10.11650/twjm/1500403962

Mecheri, S. (2017). Bishop's property and the helton class of operators. *Acta Scientiarum Mathematicarum*, *83*(1-2), 307–320. doi: 10.14232/actasm-015-374-6

Poon, Y. T. (2014). Operator equations and spectral inclusion. *Journal of Operator Theory*, *71*(2), 341–356. doi: 10.7900/jot.2012may21.1987

Salhi, J. (2015). On the single-valued extension property for operator products. *Banach Journal of Mathematical Analysis*, *9*(3), 184–195. doi: 10.15352/bjma/09-3-12

Salhi, J., & Fahlaoui, S. (2012). Local spectral properties of operator products under algebraic conditions. *Mediterranean Journal of Mathematics*, *9*(2), 349–362. doi: 10.1007/s00009-011-0123-z

Vrbóvá, P. (2000). On local spectral properties of operators in banach spaces. *Czechoslovak Mathematical Journal*, *50*(4), 801–816. doi: 10.1023/A:1022474209142

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