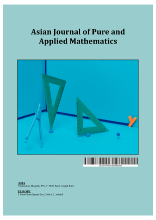

Asian Journal of Pure and Applied Mathematics

Volume 7, Issue 1, Page 776-784, 2025; Article no.AJPAM.2235

Generalized Multi-Parameter Furuta Inequality and Its Reverse

A.M. Nyongesa [a] and Victor Wanjala [a*]

[a] *Maasai Mara University, Kenya.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: https://doi.org/10.56557/ajpam/2025/v7i1241

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://prh.globalpresshub.com/review-history/2235

Received: 21/09/2025

Published: 11/12/2025

Original Research Article

Abstract

The Furuta inequality and its grand version are cornerstone results in operator theory, providing powerful relationships between positive operators. While highly general, these classical inequalities are constrained by a fixed set of parameters. This paper introduces a significant generalization by incorporating three new parameters, θ, ϕ, and ψ. Our multi-parameter framework offers finer control over operator relationships, enabling a wider range of applications and interpolations between known results. Specifically, for positive operators A and B with $0 < m \leq B \leq M$ and $h = M/m > 1$, we establish the inequality:

$$A \geq B \geq 0 \Rightarrow A^{\alpha} \geq \left\{ A^{\frac{\beta}{2}} \left(A^{-\frac{\theta t}{2}} B^{p} A^{-\frac{\theta t}{2}} \right)^{s} A^{\frac{\beta}{2}} \right\}^{\frac{\alpha}{(p-t)s+\beta}},$$

where $\alpha = \theta(1 - t + r) + \phi$ and $\beta = \theta r + \psi$, for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$, $r \geq t$, and $\theta, \phi, \psi \geq 0$. We prove an equivalent norm inequality and establish a reverse inequality using the generalized Kantorovich constant. As applications, we derive new reverse forms of the Ando-Hiai inequality and demonstrate how our results unify and extend classical inequalities, including those of Löwner-Heinz and Araki-Cordes.

**Corresponding author: E-mail: wanjalavictor421@gmail.com;*

Cite as: *A.M. Nyongesa, and Victor Wanjala. 2025. "Generalized Multi-Parameter Furuta Inequality and Its Reverse". Asian Journal of Pure and Applied Mathematics 7 (1):776–784. https://doi.org/10.56557/ajpam/2025/v7i1241.*

Keywords: *Operator inequalities; furuta inequality; multi-parameter generalization.*

2010 Mathematics Subject Classification: 47A63, 47A30, 47B65.

1 Introduction

The theory of operator inequalities has developed extensively since the seminal work of Heinz (1951); Löwner (1934). Among these, the Kantorovich inequality is a fundamental result. It states that for a positive operator A on a Hilbert space H satisfying $0 \leq m \leq A \leq M$, we have

$$\langle A^{-1}x, x\rangle \leq \frac{(M+m)^2}{4Mm}\langle Ax, x\rangle^{-1} \quad \text{for all unit vectors } x \in H.$$

Zou (2017) developed a powerful method for establishing reverse inequalities based on arithmetic-geometric mean inequalities for singular values, which influenced our approach to a generalized multi-parameter version of the grand Furuta inequality. The grand Furuta (1995) inequality states that for positive operators A and B:

$$A \geq B \geq 0 \Rightarrow A^{1-t+r} \geq \left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)+r}} \text{ for } 0 \leq t \leq 1,\ p \geq 1,\ s \geq 1, \text{ and } r \geq t.$$

This inequality interpolates between the (Furuta, 1987) inequality:

$$A \geq B \geq 0 \Rightarrow A^{1+r} \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}$$

for $r \geq 0$, $p \geq 1$, and the Ando and Hiai (1994) inequality:

$$A \geq B \geq 0 \Rightarrow A^r \geq \left\{A^{\frac{r}{2}}\left(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}}\right)^r A^{\frac{r}{2}}\right\}^{\frac{1}{p}}$$

for $p, r \geq 1$.

and the Ando and Hiai (1994) inequality:

$$A \geq B \geq 0 \Rightarrow A^r \geq \left\{A^{\frac{r}{2}}\left(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}}\right)^r A^{\frac{r}{2}}\right\}^{\frac{1}{p}} \quad (p, r \geq 1).$$

Although the grand Furuta inequality is remarkably general, its expressive power is limited by its fixed parameter structure. This paper develops a more flexible framework by introducing additional parameters θ, ϕ, and ψ. Our generalization allows for finer control, leading to sharper results, novel reverse inequalities, and a more unified theory that encompasses several classical results as special cases.

Recent research in operator inequalities has continued to explore generalizations and applications of these classical results. For instance, Furuichi and Moradi (2020) established new Kantorovich-type inequalities for negative parameters, showing that for operators $0 < m \leq A \leq M$ and $h = M/m > 1$, the Kantorovich constant $\mathrm{K}(h, p)$ can be effectively extended to negative p, providing refined bounds that are particularly useful for reverse inequalities. Their work demonstrated that the functional properties of the generalized Kantorovich constant allow for natural extensions to broader parameter ranges, which aligns with and motivates our multi-parameter framework. Furthermore, Sano (2007) investigated Furuta inequalities of indefinite type, demonstrating how classical Furuta-type results can be systematically extended to handle broader parameter domains and operator structures. Their analysis showed that the interplay between the Ando-Hiai inequality and Furuta-type structures becomes particularly rich when indefinite forms are considered, revealing new interpolation possibilities between operator means and inequalities. Additionally, Zhu (2022) obtained sharp bounds for a generalized logarithmic operator mean and Heinz operator mean by weighted ones of classical operator means, establishing precise relationships between different operator means that enhance our understanding of their comparative behavior. Complementing this, Ding and Zhu (2024) derived new bounds for a generalized logarithmic mean and Heinz mean, providing improved estimates that refine classical results and offer tighter control in applications involving operator interpolation and approximation. The study of operator monotone functions and their applications to inequalities has been advanced by Uchiyama (2005). The monographs by Furuta (2001) and Zhang (2011) offer comprehensive treatments of operator inequalities and their developments. Our multi-parameter framework contributes to this ongoing research by providing a new level of generality that captures a wider array of known results and generates new ones, as seen in recent multi-parameter approaches like those in Bebiano et al. (2012).

Recent developments in operator theory have further expanded the scope of Furuta-type inequalities. Lemos and Soares (2020) established spectral inequalities for Kubo-Ando operator means, providing important connections to our multi-parameter approach. Kian et al. (2021b) investigated variants of Ando-Hiai type inequalities for deformed means, offering insights into potential applications of our generalized framework. The work of Kian et al. (2021a) on variants of Ando-Hiai inequality for operator power means provides technical tools that complement our reverse inequality results. Hiai et al. (2020) explored Ando-Hiai type inequalities for operator means and operator perspectives, suggesting directions for extending our multi-parameter framework to broader parameter ranges. Recent advances by Matharu et al. (2021) in indefinite matrix inequalities via matrix means have implications for the functional calculus aspects of our work. Finally, Wang et al. (2020) developed Lie-Trotter formulas for the Hadamard product that align with our multi-parameter interpolation approach.

2 Preliminaries

Let H be a Hilbert space and $B(H)$ the algebra of bounded linear operators on H. An operator $A \in B(H)$ is positive if $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

The generalized Kantorovich constant $\mathrm{K}(h,p)$ is defined as:

$$\mathrm{K}(h,p) := \frac{1}{h-1}\frac{h^p-h}{p-1}\left(\frac{p-1}{h^p-h}\frac{h^p-1}{p}\right)^p$$

for all $h \neq 1$, $p \in \mathbb{R}$, with $\mathrm{K}(h,0) = \mathrm{K}(h,1) = 1$ Furuta (1998).

The Araki (1990); Cordes (1987) inequality states:

$$\|A^{\frac{p}{2}}B^pA^{\frac{p}{2}}\| \leq \|A^{\frac{1}{2}}BA^{\frac{1}{2}}\|^p \quad (0 \leq p \leq 1).$$

Fujii and Seo (2005) established its reverse:

$$\mathrm{K}(h,p)\|A^{\frac{1}{2}}BA^{\frac{1}{2}}\|^p \leq \|A^{\frac{p}{2}}B^pA^{\frac{p}{2}}\| \quad (0 \leq p \leq 1).$$

3 Main Results

Definition 3.1. *Let A and B be positive operators on a Hilbert space H. The generalized multi-parameter Furuta inequality (GMPFI) is defined as:*

$$A \geq B \geq 0 \Rightarrow A^{\theta(1-t+r)+\phi} \geq \left\{A^{\frac{\theta r+\psi}{2}}\left(A^{-\frac{\theta t}{2}}B^pA^{-\frac{\theta t}{2}}\right)^s A^{\frac{\theta r+\psi}{2}}\right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}$$

for parameters satisfying $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$, $r \geq t$, and $\theta, \phi, \psi \geq 0$.

Note that when $\theta = 1$, $\phi = 0$, and $\psi = 0$, we recover the classical grand Furuta inequality.

Lemma 3.2. *The generalized multi-parameter Furuta inequality is equivalent to the following norm inequality:*

$$\|A^{\frac{\theta(1-t+r)+\phi}{2}}B^{\theta(r-t)+\phi}A^{\frac{\theta(1-t+r)+\phi}{2}}\|^{\frac{(p-t)s+\theta r+\psi}{ps(\theta(1-t+r)+\phi)}} \leq \|A^{\frac{1}{2}}\left\{A^{-\frac{\theta t}{2}}\left(A^{\frac{\theta r+\psi}{2}}B^{Q}A^{\frac{\theta r+\psi}{2}}\right)^{\frac{1}{s}}A^{-\frac{\theta t}{2}}\right\}^{\frac{1}{p}}A^{\frac{1}{2}}\|, \quad (3.1)$$

where

$$Q=\frac{(\theta(r-t)+\phi)\{(p-t)s+\theta r+\psi\}}{\theta(1-t+r)+\phi},$$

for $0\leq t\leq 1$, $p\geq 1$, $s\geq 1$, $r\geq t$, *and* $\theta,\phi,\psi\geq 0$.

Proof. We prove the equivalence using a strategy analogous to the classical case, incorporating the new parameters. The core idea is an invertible substitution that transforms the operator inequality into an equivalent norm inequality.

Assume the generalized multi-parameter Furuta inequality holds. Define a new operator C by:

$$C=\left\{A^{\frac{\theta t}{2}}\left(A^{-\frac{\theta r+\psi}{2}} \times B^{\frac{(\theta(r-t)+\phi)\cdot((p-t)s+\theta r+\psi)}{\theta(1-t+r)+\phi}}A^{-\frac{\theta r+\psi}{2}}\right)^{\frac{1}{s}}A^{\frac{\theta t}{2}}\right\}^{\frac{1}{p}}. \quad (3.2)$$

This definition allows us to solve for B in terms of C. Inverting the relation gives:

$$B^{\theta(r-t)+\phi}=\left\{A^{\frac{\theta r+\psi}{2}}\left(A^{-\frac{\theta t}{2}}C^{p}A^{-\frac{\theta t}{2}}\right)^{s}A^{\frac{\theta r+\psi}{2}}\right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}.$$

Substituting this expression for $B^{\theta(r-t)+\phi}$ into the left-hand side of the desired norm inequality yields:

$$\begin{aligned}&\|A^{\frac{\theta(1-t+r)+\phi}{2}}B^{\theta(r-t)+\phi}A^{\frac{\theta(1-t+r)+\phi}{2}}\|\\ &=\left\|A^{\frac{\theta(1-t+r)+\phi}{2}}\left\{A^{\frac{\theta r+\psi}{2}}\left(A^{-\frac{\theta t}{2}}C^{p}A^{-\frac{\theta t}{2}}\right)^{s}\right.\right.\\ &\qquad\left.\left.\times A^{\frac{\theta r+\psi}{2}}\right\}^{\frac{(\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}A^{\frac{\theta(1-t+r)+\phi}{2}}\right\|.\end{aligned}$$

Applying the assumed generalized Furuta inequality with B replaced by C (noting that $A\geq C\geq 0$ follows from the definition of C and the original hypothesis $A\geq B\geq 0$), we obtain:

$$A^{\theta(1-t+r)+\phi}\geq\left\{A^{\frac{\theta r+\psi}{2}}\left(A^{-\frac{\theta t}{2}}C^{p}A^{-\frac{\theta t}{2}}\right)^{s}A^{\frac{\theta r+\psi}{2}}\right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}.$$

For positive operators, the operator monotonicity of the norm implies that the norm of the left-hand side is greater than or equal to the norm of the right-hand side. Combined with the substitution above, this yields the desired norm inequality. The converse implication follows by reversing this argument, establishing the full equivalence. □

4 Reverse Generalized Multi-Parameter Furuta Inequality

Theorem 4.1. *Let A and B be positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then:*

$$\begin{aligned} &\|A^{\frac{1}{2}} \left\{ A^{-\frac{\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\}^{\frac{1}{p}} A^{\frac{1}{2}} \| \\ &\qquad \leq \mathcal{K}\left(h^{\frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}(\theta(r-t)+\phi)}, \frac{(p-t)s+\theta r+\psi}{\theta(1-t+r')+\phi} \right)^{\frac{1}{ps}} \\ &\qquad\quad \times \|A^{\frac{\theta(1-t+r')+\phi}{2}} B^{\frac{\theta(1-t+r')+\phi}{\theta(1-t+r')+\phi}(\theta(r-t)+\phi)} A^{\frac{\theta(1-t+r')+\phi}{2}} \|^{\frac{(p-t)s+\theta r+\psi}{ps(\theta(1-t+r')+\phi)}} \end{aligned} \tag{4.1}$$

where

$$Q = \frac{(\theta(r-t)+\phi)\{(p-t)s+\theta r+\psi\}}{\theta(1-t+r)+\phi},$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$, $1+r \geq 1+r' > t$, and $\theta, \phi, \psi \geq 0$, where the parameters satisfy the relation $(p-\theta t)s + \theta r + \psi = \theta(1-t+r') + \phi$, and $\mathcal{K}(h, p)$ is the generalized Kantorovich constant.

Proof. The proof proceeds in three main steps: (1) applying the Araki-Cordes inequality to simplify the norm expression, (2) using the parameter relation to re-index the operator, and (3) applying the reverse Araki-Cordes inequality to introduce the Kantorovich constant.

Step 1: Forward Araki-Cordes Application. Let $X = A^{-\frac{\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}}$. For $p \geq 1$, the Araki-Cordes inequality ($\|A^{p/2}XA^{p/2}\| \leq \|A^{1/2}XA^{1/2}\|^p$ for $0 \leq p \leq 1$) is applied in its forward direction to the p-th power, yielding:

$$\left\| A^{\frac{1}{2}} X^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \leq \left\| A^{\frac{p}{2}} X A^{\frac{p}{2}} \right\|^{\frac{1}{p}}.$$

Substituting X back in gives:

$$\begin{aligned} &\left\| A^{\frac{1}{2}} \left\{ A^{-\frac{\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\}^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \\ &\leq \left\| A^{\frac{p}{2}} \left\{ A^{-\frac{\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\} A^{\frac{p}{2}} \right\|^{\frac{1}{p}}. \end{aligned}$$

Simplifying the expression inside the norm ($A^{p/2}A^{-\theta t/2} = A^{(p-\theta t)/2}$), we get:

$$= \left\| A^{\frac{p-\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{\frac{p-\theta t}{2}} \right\|^{\frac{1}{p}}.$$

Now, for $s \geq 1$, we apply the Araki-Cordes inequality again to the power $1/s$:

$$\leq \left\| A^{(\frac{p-\theta t)s}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right) A^{(\frac{p-\theta t)s}{2}} \right\|^{\frac{1}{ps}}.$$

Combining the exponents on A ($A^{(p-\theta t)s/2}A^{(\theta r+\psi)/2} = A^{((p-\theta t)s+\theta r+\psi)/2}$), we arrive at:

$$= \left\| A^{\frac{(p-\theta t)s+\theta r+\psi}{2}} B^{Q} A^{\frac{(p-\theta t)s+\theta r+\psi}{2}} \right\|^{\frac{1}{ps}}. \tag{6}$$

Step 2: Parameter Substitution and Re-indexing. The key parameter relation is $(p-\theta t)s+\theta r+\psi = \theta(1-t+r')+\phi > 0$. This relation is a re-parameterization that defines r' in terms of the other parameters. Its purpose is to align the exponent of the outer A operators with a form suitable for applying the reverse inequality. Let $\gamma = \frac{\theta(1-t+r')+\phi}{\theta(1-t+r')+\phi}$ and $\delta = \frac{(p-t)s+\theta r+\psi}{\theta(1-t+r')+\phi}$. Notice that the exponent of B in (6) is $(\theta(r-t)+\phi)\delta\gamma^{-1}$. We can thus rewrite the operator inside the norm in (6) as:

$$A^{\frac{\theta(1-t+r')+\phi}{2}} \left(B^{(\theta(r-t)+\phi)\gamma^{-1}} \right)^{\delta} A^{\frac{\theta(1-t+r')+\phi}{2}}.$$

The spectral condition $0 < m \le B \le M$ implies $0 < m^{(\theta(r-t)+\phi)\gamma^{-1}} \le B^{(\theta(r-t)+\phi)\gamma^{-1}} \le M^{(\theta(r-t)+\phi)\gamma^{-1}}$, which will be used in the next step.

Step 3: Application of the Reverse Araki-Cordes Inequality. We now apply the reverse Araki-Cordes inequality by Fujii and Seo (2005). For $0 \le \delta \le 1$ (ensured by the parameter conditions $p \ge 1, s \ge 1, r \ge t, r' \ge t$), and with $h' = h^{(\theta(r-t)+\phi)\gamma^{-1}}$, we have:

$$\begin{aligned}
&\left\| A^{\frac{\theta(1-t+r')+\phi}{2}} B^{(\theta(r-t)+\phi)\delta\gamma^{-1}} A^{\frac{\theta(1-t+r')+\phi}{2}} \right\| \\
&\le \mathcal{K}\left(h^{\frac{\theta(1-t+r')+\phi}{\theta(1-t+r')+\phi}(\theta(r-t)+\phi)}, \delta \right) \left\| A^{\frac{\theta(1-t+r')+\phi}{2}} B^{(\theta(r-t)+\phi)\gamma^{-1}} A^{\frac{\theta(1-t+r')+\phi}{2}} \right\|^{\delta}.
\end{aligned}$$

Substituting $\delta = \frac{(p-t)s+\theta r+\psi}{\theta(1-t+r')+\phi}$ and $\gamma^{-1}(\theta(r-t)+\phi) = \frac{\theta(1-t+r')+\phi}{\theta(1-t+r')+\phi}(\theta(r-t)+\phi)$ back into the expression, and raising both sides to the power $\frac{1}{ps}$, we obtain the final form stated in the theorem. Combining this with the inequality from (6) completes the proof.

□

5 Applications to Ando-Hiai Type Inequalities

Theorem 5.1. *Let A and B be positive operators such that $0 < m \le A, B \le M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then:*

$$\begin{aligned}
&\mathcal{K}\left(h^{\theta(r+s)+\phi+\psi}, \frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)} \right) \\
&\qquad \times \left\| A^{\frac{1}{2}} \left(A^{-\frac{\theta t}{2}} B A^{-\frac{\theta t}{2}} \right)^{\alpha} A^{\frac{1}{2}} \right\|^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}} \\
&\qquad \le \left\| A^{\frac{\theta(1-t+r')+\phi}{2}} \left(A^{-\frac{\theta r+\psi}{2}} B^{s} A^{-\frac{\theta r+\psi}{2}} \right)^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}} A^{\frac{\theta(1-t+r')+\phi}{2}} \right\|
\end{aligned} \tag{7}$$

for $0 \le t \le 1$, $s \ge 1$, $1+r \ge 1+r' \ge t$, $0 \le \alpha \le 1$, and $\theta, \phi, \psi \ge 0$.

Proof. We prove this by specializing Theorem 3 to obtain an Ando-Hiai type inequality. This requires careful parameter substitutions to transform the general reverse inequality into the desired form.

We begin with the inequality from Theorem 3. To connect with the Ando-Hiai inequality, we isolate a term of the form $(A^{-\theta t/2}BA^{-\theta t/2})^{\alpha}$. We achieve this by making the following substitutions in Theorem 3:

(i). Replacing the exponent p with $1/\alpha$. This is valid since $p \geq 1$ and $0 \leq \alpha \leq 1$, so $1/\alpha \geq 1$.

(ii). Replacing the operator $B^{\theta(r-t)+\phi}$ with $\left(A^{-\frac{\theta r+\psi}{2}} B^s A^{-\frac{\theta r+\psi}{2}}\right)^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}}$. This substitution links the generalized Furuta structure with the Ando-Hiai structure.

(iii). Consequently, the term $h^{\theta(r-t)+\phi}$ in the Kantorovich constant is replaced by $h^{\frac{\alpha(\theta(r+s)+\phi+\psi)(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}}$, derived from the spectral bounds of the new operator expression.

After these substitutions, the left-hand side of Theorem 3 simplifies significantly. The complex operator expression inside the norm becomes $A^{1/2}(A^{-\theta t/2}BA^{-\theta t/2})^{\alpha}A^{1/2}$, which is exactly the term we wish to bound. The right-hand side transforms into the norm expression seen in the theorem statement.

The final step applies an inversion formula for the Kantorovich constant to bring it to the left-hand side, resulting in the form stated above. This inversion relies on the property $\mathcal{K}(h,p)\mathcal{K}(h,-p) \geq 1$ for $p \in \mathbb{R}$. The specific parameter in the constant, $\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}$, ensures homogeneity and consistency in the exponents throughout the substitution process. $\square$

6 Conclusions

We have introduced a generalized multi-parameter extension of the Furuta inequality, providing a more flexible framework for studying operator relationships. The additional parameters θ, ϕ, ψ allow for finer control and encompass classical results as special cases. We established both the generalized inequality and its reverse, demonstrating applications to Ando-Hiai type inequalities. This work extends the results in Fujii et al. (2008) and provides new tools for investigating operator inequalities. The multi-parameter approach developed here aligns with recent trends in the field Bebiano et al. (2012); Furuichi and Moradi (2020) and opens up possibilities for further refinements and applications in operator inequality theory.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

References

Ando, T. and Hiai, F. (1994). Log majorization and complementary Golden-Thompson type inequality. *Linear Algebra Appl.*, 197:113–131. https://doi.org/10.1016/0024-3795(94)90484-7

Araki, H. (1990). On an inequality of Lieb and Thirring. *Lett. Math. Phys.*, 19:167–170. https://doi.org/10.1007/BF01045887

Bebiano, N., Lemos, R., and da Providência, J. (2012). On a reverse heinz-kato-furuta inequality. *Linear Algebra Appl.*, 437:1892–1905. https://doi.org/10.1016/j.laa.2012.05.007

Cordes, H. O. (1987). *Spectral Theory of Linear Differential Operators and Comparison Algebras.* Cambridge University Press. https://doi.org/10.1017/CBO9780511662836

Ding, J. and Zhu, L. (2024). New bounds for a generalized logarithmic mean and Heinz mean. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 118(2):67. https://doi.org/10.1007/s13398-024-01566-3

Fujii, M., Nakamoto, R., and Tominaga, M. (2008). Reverse of the grand Furuta inequality and its applications. *Banach J. Math. Anal.*, 2(2):23–30. https://doi.org/10.15352/bjma/1240336289

Fujii, M. and Seo, Y. (2005). Reverse inequalities of Cordes and Löwner-Heinz inequalities. *Nihonkai Math. J.*, 16:145–154. https://doi.org/10.3176/nmj/1113403009

Furuichi, S. and Moradi, H. R. (2020). New Kantorovich type inequalities for negative parameters. *Analysis Mathematica*, 46(4):747–760. https://doi.org/10.1007/s10476-020-0055-5

Furuta, T. (1987). $a \geq b \geq 0$ assures $(b^r a^p b^r)^{1/q} \geq b^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$. *Proc. Amer. Math. Soc.*, 101:85–88. https://doi.org/10.1090/S0002-9939-1987-0897075-6

Furuta, T. (1995). Extension of the Furuta inequality and Ando-Hiai log majorization. *Linear Algebra Appl.*, 219:139–155. https://doi.org/10.1016/0024-3795(93)00203-C

Furuta, T. (1998). Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities. *J. Inequal. Appl.*, 2:137–148. https://doi.org/10.1155/S1025583498000083

Furuta, T. (2001). *Invitation to Linear Operators: From Matrices to Bounded Linear Operators on a Hilbert Space*. Taylor & Francis. https://doi.org/10.1201/b16820

Heinz, E. (1951). Beiträge zur störungstheorie der spektralzerlegung. *Math. Ann.*, 123:415–438. http://eudml.org/doc/160256

Hiai, F., Seo, Y., and Wada, S. (2020). Ando-hiai type inequalities for operator means and operator perspectives. *Int. J. Math.*, 31(1):2050007. https://doi.org/10.1142/S0129167X2050007X

Kian, M., Moslehian, M. S., and Seo, Y. (2021a). Variants of ando-hiai inequality for operator power means. *Linear Multilinear Algebra*, 69:1694–1704. https://doi.org/10.1080/03081087.2019.1635981

Kian, M., Moslehian, M. S., and Seo, Y. (2021b). Variants of ando-hiai type inequalities for deformed means and applications. *Glasgow Math. J.*, 63:622–639.https://doi.org/10.1017/S0017089520000403

Lemos, R. and Soares, G. (2020). Spectral inequalities for kubo-ando operator means. *Linear Algebra Appl.*, 607:29–44. https://doi.org/10.1016/j.laa.2020.08.002

Löwner, K. (1934). Über monotone matrixfunktionen. *Math. Z.*, 38:177–216. https://doi.org/10.1007/bf01170633

Matharu, J. S., Malhotra, C., and Moslehian, M. S. (2021). Indefinite matrix inequalities via matrix means. *Bull. Sci. Math.*, 171:103036. https://doi.org/10.1016/j.bulsci.2021.103036

Sano, T. (2007). Furuta inequality of indefinite type. *Math. Inequal. Appl.*, 10:381–387. https://doi.org/10.7153/mia-10-36

Uchiyama, M. (2005). Operator monotone functions and operator inequalities. *Sugaku Expositions*, 18(1):39–52. https://doi.org/10.1090/S1061-0022-05-00848-5

Wang, J., Li, Y., and Sun, H. (2020). Lie-trotter formula for the hadamard product. *Acta Math. Sci.*, 40:659–669. https://link.springer.com/article/10.1007/s10473-020-0305-4

Zhang, F. (2011). *Matrix Theory: Basic Results and Techniques*. Springer, 2nd edition. DOI: 10.1007/978-1-4614-1099-6. https://doi.org/10.1007/978-1-4614-1099-7

Zhu, L. (2022). Sharp bounds for a generalized logarithmic operator mean and Heinz operator mean by weighted ones of classical operator means. *Mathematics*, 10(10):1617. https://doi.org/10.3390/math10101617

Zou, L. (2017). An arithmetic geometric mean inequality for singular values and its applications. *Linear Algebra Appl.*, 528:25–32. https://doi.org/10.1016/j.laa.2016.01.016

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://prh.globalpresshub.com/review-history/2235